THE

FIRST SIX BOOKS

OF THE

ELEMENTS OF EUCLID,

WITH NOTES.

TENTH EDITION.

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TO THE READER.

In preparing this edition, designed solely for the use of the Undergraduates in the University of Dublin, it was the object of the Editor to retain the original strictness of the demonstrations as given by Euclid, and at the same time to throw them into the most simple form. For this purpose the syllogistic mode of argument, in which all the direct proofs had usually been drawn up, has been abandoned; and the arrangement approved of by Locke adopted in its stead. But the theory of Proportion in the fifth book has been entirely altered, for reasons assigned in the notes. To defend the definition of proportional quantities given by Euclid, it has been asserted, that there is no other principle from which the doctrine can be demonstrated; but that defence, it is hoped, must now be abandoned, as the demonstrations here given are derived from a more simple and natural definition.

In overcoming the difficulties of establishing this new principle, which will appear considerable to any person who examines the propositions of the fifth book, the Editor has to acknowledge the assistance afforded by his predecessor in the Professorship of Mathematics, the venerable Dr. Murray, who communicated to him the elegant and comparatively easy demonstrations of the 20th and 38th propositions; but such was his averseness from being known, that this obligation could not be acknowledged till after his death, which unfortunately followed but too soon his much desired and universally applauded promotion to the Provostship.

The corollaries and scholia have been selected from the best commentators, with anxious care to retain whatever was useful, and to avoid encumbering the work with any thing superfluous. In the notes the motives for any changes deserving of notice are briefly assigned.

THOMAS ELRINGTON.

Trin. Coll. Dub.

N. B. This work has been translated from the last Latin Edition, with the consent of the Editor, and at the request of the Masters of several English schools in Ireland.

THE

ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

1. A point is that which hath no parts. See Notes.
2. A line is length without breadth.
3. The extremities of a line are points.
4. A right line is that which lies evenly between its extreme points. Plate 1. Fig. 1.
5. A surface is that which hath only length and breadth.
6. The extremities of a surface are lines.
7. A plane surface is that which lies evenly between its extreme right lines. See N.
8. A rectilineal angle is the inclination of two right lines to one another, which meet together, but are not in the same right line. See N. Fig. 2.
9. The sides of an angle are the lines which form the angle. See N.
10. The vertex of an angle is the point in which the sides meet one another.

An angle is expressed either by one letter placed at the vertex; or by three letters, of which the middle one is at the vertex, the others any where along the sides.

11. When a right line standing on another makes the adjacent angles (ABC and ABD) equal to one another, each of these angles is called a right angle; Fig. 3.

and the right line, which stands on the other, is called a perpendicular to it.

Fig. 4. 12. The angle (ABC), which is greater than a right angle, is called obtuse.

Fig. 4. See N. 13. The angle (ABD), which is less than a right angle, is called acute.

14. A plane figure is a plane surface, which is bounded on all sides by one or more lines.

Fig. 5. 15. A circle is a plane figure bounded by one line, which is called the circumference, and is such that all right lines, drawn from a certain point within the figure to the circumference, are equal to one another.

16. And this point is called the centre of the circle.

See N. 17. A diameter of a circle is a right line drawn through the centre, and terminated both ways by the circumference.

18. A radius of a circle is a right line drawn from the centre to the circumference.

19. A semicircle is the figure contained by a diameter, and the part of the circumference cut off by the diameter.

20. A rectilineal figure is a plane surface bounded by right lines.

21. A triangle is a rectilineal figure bounded by three right lines.

Fig. 6. 22. An equilateral triangle is that which has three equal sides.

Fig. 7. 23. An isosceles triangle is that which has two sides equal.

Fig. 8. 24. A scalene triangle is that which has three unequal sides.

Fig. 8. 25. A right-angled triangle is that in which one of the angles is right.

Fig. 9. 26. An obtuse-angled triangle is that in which one of the angles is obtuse.

Fig. 7. 27. An acute-angled triangle is that in which the three angles are acute.

Fig. 10. 28. Parallel right lines are those which, lying in the same plane, never meet on either side, though indefinitely produced.

29. A quadrilateral figure is a rectilineal figure, which is bounded by four right lines.

30. A parallelogram is a quadrilateral figure, whose opposite sides are parallel. Fig. 11.

31. A square is a quadrilateral figure, which has all its sides equal, and all its angles right angles.

32. Rectilineal figures, which have more than four sides, are called polygons.

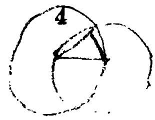
POSTULATES.

Let it be granted,

1. That a right line may be drawn from any one point to any other.
2. That a terminated right line may be produced to any length in a right line.
3. That a circle may be described from any centre, at any distance from that centre. See N.

AXIOMS.

1. Things, which are equal to the same, are equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be taken from equals, the remainders are equal.
4. If equals be added to unequals, the wholes are unequal.
5. If equals be taken from unequals, the remainders are unequal.
6. Things, which are double of the same or of equals, are equal to one another.
7. Things, which are halves of the same or of equals, are equal to one another.
8. Magnitudes, which coincide with one another, are equal to one another.
9. The whole is greater than its part.
10. Two right lines cannot inclose a space.
11. All right angles are equal to one another.
12. If a right line meet two right lines, so as to make the two internal angles on the same side of it taken together less than two right angles; these right lines, being continually produced, shall at length meet upon that side on which are the angles, which are less than two right angles. See N.



PROPOSITION I. PROBLEM.

Fig. 12. See N. *To describe an equilateral triangle upon a given finite right line (AB).*

From the centre A, with the radius AB, describe the
(1) Post. 3. circle BCD (1), and from the centre B, with the radius BA, describe the circle ACE. From the point of intersection C, draw the right lines CA and CB to the ex-
(2) Post. 1. tremities of the given right line (2).

It is evident that ACB is a triangle constructed upon the given line; but it is also equilateral: for the right line AC is equal to AB, as they are radii of the same circle
(3) Def. 13. DCB (3); and the right line BC is equal to BA, as they are radii of the same circle ACE. Since then both the lines AC and BC are equal to the same AB, they must
(4) Ax. 1. be equal to one another (4), and therefore the triangle ACB is equilateral.

Schol. Draw the lines AG and GB; and in the same manner it can be demonstrated that the triangle AGB is equilateral.

PROP. II. PROB.

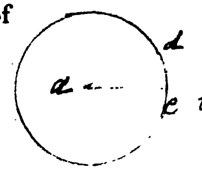
Fig. 13. See N. *From a given point (A) to draw a right line equal to a given finite right line (BC).*

From the given point A draw a right line AB to
(1) Post. 1. either extremity B of the given line (1). Upon AB
(2) Prop. 1. construct an equilateral triangle ADB (2). From the
(3) Post. 3. centre B, with the radius BC, describe a circle GCF (3), and produce the right line DB, until it meets the circumference in G. From the centre D, with the radius DG, describe the circle GLO; and produce the right line DA, until it meets the circumference in L. The right line AL is equal to the given line BC.

For the right line DL is equal to DG, because they
(4) Def. 15. are radii of the same circle GLO (4); and if the equals
(5) Construct. DA and DB (5) be taken away from both, the re-
(6) Ax. 3. mainders AL and BG are also equal (6); but BG is equal to BC, as they are radii of the same circle GCF; therefore the right lines AL and BC, which are equal

to the same BG, are equal to one another (7). Therefore from the given point A a right line AL has been drawn, equal to the given right line BC. (7) Ax. 1.

Schol. The position of the right line AL varies according to the extremity of the given right line from which it is drawn, and also according to the side of that line on which the triangle is constructed.



PROP. III. PROB.

From the greater of two given right lines (AB and CF) to cut off a part equal to the less. Fig. 14. See N.

From either extremity A of the greater line draw AD, equal to CF the less of the given right lines (1). (1) Prop. 2.
From the centre A, with the radius AD, describe a circle which shall cut off AE equal to AD (2), and therefore also equal to the given right line CF (3). (2) Def. 15. (3) Constr. & Ax. 1.

PROP. IV. THEOREM.

If two triangles (EDF, ABC) have two sides of the one respectively equal to two sides of the other (ED and DF to AB and BC) and the angles contained by those sides also equal to one another (D to B); their bases (EF and AC) are equal, and the angles at the bases, which are opposite to the equal sides, are equal (E to A and F to C); and also the triangles themselves. Fig. 15. See N.

For if the triangle EDF be so applied to the triangle ABC, that the point D may be on B, and the side DE on BA, and that DF and BC may lie at the same side, the point E must coincide with A, because the sides DE and BA are equal: and because the angles D and B are equal, the side DF must fall on BC; and because the side DF is equal to BC, the point F must coincide with C.

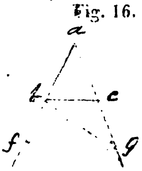
But as the points E and F coincide with the points A and C, the right lines EF and AC must coincide (1), and therefore the bases EF and AC are equal (2). (1) Ax. 10. (2) Ax. 8.

And as the sides of the angles E and F coincide with the sides of the angles A and C, the angles themselves (3) Ax. 8. must coincide, and are therefore equal (3).

And as the right lines, which bound the triangle EDF, coincide with the right lines which bound the triangle ABC, the triangles themselves must coincide, and are therefore equal (3).

PROP. V. THEOR.

Fig. 16. *In an isosceles triangle (BAC) the angles at the base (ABC and ACB) are equal to one another; and if the equal sides be produced, the angles below the base (FBC and GCB) shall also be equal.*



Take any point F in the side produced, cut off AG (1) Prop. 3. equal to AF (1), and join FC, GB.

In the triangles FAC and GAB, the sides FA and (2) Constr. & Hypoth. AC are equal to the sides GA and AB (2), and the angle A is common to both, therefore the angle ACF is equal to ABG, and the angle AFC to AGB; and the (3) Prop. 4. side FC is equal to the side GB (3). Therefore in the triangles BFC, CGB, the angle BFC is equal to the angle CGB, and the side FC is equal to BG, and taking away the equals AB and AC from the equals AF and AG, the side BF is equal to the side CG, therefore the angle FBC is equal to the angle GCB (3); but these are the angles below the base BC.

And in the same triangles, the angle FCB is equal to the angle GBC (3), and taking away these from the equals FCA and GBA, the remaining angles ACB and (4) Ax. 3. ABC are equal (4), but these are the angles at the base BC of the given triangle.

Cor. Hence every equilateral triangle is also equiangular; for whatever side is taken for the base, the angles adjacent to it are equal, since they are opposite to equal sides.

PROP. VI. THEOR.

If two angles (B and C) of a triangle (BAC) be equal, the sides opposite to them (AC and AB) are also equal. Fig. 17.

If the sides be not equal, let one of them AB be greater than the other, and from it cut off DB equal to AC (1), and join CD. (1) Prop. 3.

Then in the triangles DBC, and ACB, the sides DB, BC, are equal to the sides AC, CB; and the angles DBC and ACB are also equal (2); therefore the triangles themselves, DBC and ACB, are equal (3), a part equal to the whole, which is absurd: therefore neither of the sides AB or AC is greater than the other; they are therefore equal to one another. (2) Hypoth. (3) Prop. 4.

Cor. Hence every equiangular triangle is also equilateral; for whatever side is taken for the base, the angles adjacent to it are equal, and therefore the sides which subtend them.

PROP. VII. THEOR.

On the same right line (AB) and on the same side of it, there cannot be constructed two triangles (ACB, ADB), whose conterminous sides (AC and AD, BC and BD) are equal. Fig. 18, 19, and 20.

If it be possible, let the two triangles be constructed: and first let the vertex of each of the triangles be without the other triangle, and draw CD. Fig. 18.

Because the sides AD and AC of the triangle CAD are equal (1), the angles ACD and ADC are equal (2), but ACD is greater than BCD (3), therefore ADC is greater than BCD; but the angle BDC is greater than ADC (3), and therefore BDC is greater than BCD: but in the triangle CBD, the sides BC and BD are equal (4), therefore the angles BDC and BCD are equal (5), but the angle BDC has been proved to be greater than BCD, which is absurd. Therefore the triangles constructed upon the same (1) Hypoth. (2) Prop. 5. (3) Ax. 9. (4) Hypoth. (5) prop. 6.

right line cannot have their conterminous sides equal, when the vertex of each of the triangles is without the other.

Fig. 19. Secondly. Let the vertex D of one triangle be within the other: produce the sides AC and AD, and join CD.

Because the sides AC and AD of the triangle CAD are equal (6), the angles ECD and FDC are equal (7); but the angle BDC is greater than FDC (8), and therefore greater than ECD; but ECD is greater than BCD (8), and therefore BDC is greater than BCD: but in the triangle CBD, the sides BC and BD are equal (6), therefore the angles BDC and BCD are equal (7); but the angle BDC has been proved to be greater than BCD, which is absurd. Therefore triangles constructed upon the same right line cannot have their conterminous sides equal, if the vertex of one of them be within the other.

(6) Hypoth.
(7) Prop. 5.
(8) Ax. 9.

Fig. 20. Thirdly. Let the vertex D of one triangle be on the side of the other AC; and it is evident that the sides AC and AD are not equal. Therefore in no case can two triangles, whose conterminous sides are equal, be constructed at the same side of a given line.

PROP. VIII. THEOR.

Fig. 21. See N.

If two triangles (ABC and EFD) have two sides of the one respectively equal to two sides of the other (AB to EF and BC to FD), and also have the base (AC) equal to the base (ED), then the angles (B and F) contained by the equal sides are equal.

For if the equal bases AC and ED be so applied to one another, that the equal sides AB and EF, CB and DF may be conterminous, the vertex B must fall upon F (1), and the equal sides AB and EF, CB and DF must coincide (2): therefore the angles B and F must coincide, and are therefore equal (3).

(1) Prop. 7.
(2) Ax. 10.
(3) Ax. 8.

Schol. It is evident that the remaining angles A and E, C and D, opposite to the equal sides, are equal and also that the triangles themselves are equal.

PROP. IX. PROB.

To bisect a given rectilineal angle (BAC). Fig. 22.

Take any point D in the side AB, and from AC cut off AE equal to AD (1); draw DE, and upon it describe an equilateral triangle DFE (2), at the side remote from A. The right line, joining the points A and F, bisects the given angle BAC. (1) Prop. 3. (2) Prop. 1.

Because the sides AD and AE are equal (3), and the side AF is common to the triangles FAD and FAE, and the base FD is also equal to FE (3); the angles DAF and EAF are equal (4), and therefore the right line AF bisects the given angle. (3) Constr. (4) Prop. 8.

Cor. By this proposition an angle can also be divided into 4 parts, 8, 16, &c. &c. by bisecting again each part.

PROP. X. PROB.

To bisect a given finite line (AB). Fig. 23.

Upon the given line AB describe an equilateral triangle ACB (1); bisect the angle ACB by the right line CD (2): this line bisects the given line in the point D. (1) Prop. 1. (2) Prop. 9.

Because the sides AC and CB are equal (3), and CD is common to the triangles ACD and BCD, and the angles ACD and BCD are also equal (3); therefore the bases AD and DB are equal (4), and the right line AB is bisected in the point D. (3) Constr. (4) Prop. 4.

PROP. XI. PROB.

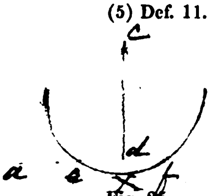
From a given point (C), in a given right line (AB), to draw a perpendicular to the given line. Fig. 24.

In the given line take any point D, and make CE equal to CD (1); upon DE describe an equilateral triangle DFE (2), draw FC, and it is perpendicular to the given line. (1) Prop. 3. (2) Prop. 1.

Because the sides DF and DC are equal to the sides

(3) Constr. EF and EC (3), and CF is common to the triangles DFC and EFC; therefore the angles DCF and ECF,
(4) Schol. opposite to the equal sides DF and EF, are equal (4), and therefore FC is perpendicular to the given right
(5) Def. 11. line AB at the point C (5).

Schol. In the same manner a perpendicular can be erected at the extremity of a given line, by first producing the line.

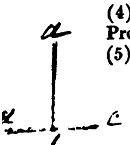


PROP. XII. PROB.

Fig. 25. See N. *To draw a perpendicular to a given indefinite right line (AB), from a point (C) given without it.*

Take any point X on the other side of the given line, and from the centre C with the radius CX describe a circle, cutting the given line in E and F. Bisect EF in
(1) Prop. 10. D (1), and draw from the given point to the point of bisection the right line CD: it is perpendicular to the given line.

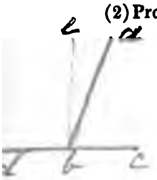
For draw CE and CF; and, in the triangles EDC and
(2) Def. 15. FDC, the sides EC and FC (2) are equal, and ED and
(3) Constr. FD (3) are equal, and CD is common; therefore the angles EDC and FDC, opposite to the equal sides EC and
(4) Schol. Prop. 8. FC, are equal (4), and therefore DC is perpendicular to
(5) Def. 11. the given line AB (5).



PROP. XIII. THEOR.

Fig. 26. See N. *When a right line (AB) standing upon another (DC) makes angles with it, they are either two right angles, or together equal to two right angles.*

If the right line AB be perpendicular to DC, the
(1) Def. 11. angles ABC and ABD are right (1). If not, draw
(2) Prop. 11. BE perpendicular to DC (2); and it is evident that the angles CBA and ABD together are equal to the angles CBE and EBD, and therefore equal to two right angles.



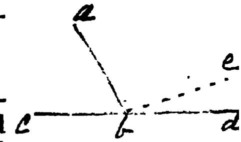
Cor. 1. If several right lines stand on the same right line at the same point, and make angles with it; all the angles taken together are equal to two right angles.

Cor. 2. Two right lines intersecting one another make angles, which, taken together, are equal to four right angles.

Cor. 3. If several right lines intersect one another in the same point, all the angles taken together are equal to four right angles.

PROP. XIV. THEOR.

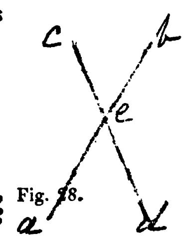
If two right lines (CB and BD), meeting another right line (AB) at the same point and at opposite sides, make angles with it, which are together equal to two right angles; those right lines (CB and BD) form one continued right line. Fig. 27.



If it be possible, let BE and not BD be the continuation of the right line CB; then the angles CBA and ABE are equal to two right angles (1): but CBA and ABD are also equal to two right angles (2); therefore CBA and ABD taken together are equal to CBA and ABE: take away from these equal quantities CBA, which is common to both, and ABE shall be equal to ABD, a part to the whole, which is absurd. Therefore BE is not the continuation of CB; and in the same manner it can be proved that no other line, except BD, is the continuation of it: therefore BD forms with BC one continued right line.

(1) Prop. 13.
(2) Hypoth.

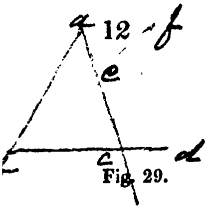
PROP. XV. THEOR.



If two right lines (AB and CD) intersect one another, the vertical angles are equal (CEA to BED, and CEB to AED). Fig. 28.

Because the right line CE stands upon the right line AB, the angle AEC, together with the angle CEB, is equal to two right angles (1); and because the right line BE stands upon the right line CD, the angle CEB, together with the angle BED, is equal to two right angles (1); therefore AEC and CEB together are equal to CEB and BED: take away the common angle CEB, and the remaining angle AEC is equal to BED (2).

(1) Prop. 13.
(2) Ax. 3.



PROP. XVI. THEOR.

Fig. 29. *If one side (BC) of a triangle (BAC) be produced, the external angle (ACD) is greater than either of the internal opposite angles (A or B).*

(1) Prop. 10. Bisect the side AC in E (1); draw BE, and produce
(2) Prop. 3. it until EF be equal to BE (2), and join FC.

The triangles CEF and AEB have the sides CE
(3) Constr. and EF equal to the sides AE and EB (3), and the
(4) Prop. 15. angle CEF equal to AEB (4); therefore the angles
(5) Prop. 4. ECF and A are equal (5), and therefore ACD is greater than A. In like manner it can be shewn that, if AC be produced, the external angle BCG is greater than the angle B, and therefore that the angle ACD, which is equal to BCG (4), is greater than the angle B.

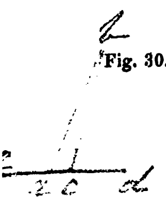


Fig. 30. Cor. 1. If, from any point B, two right lines be drawn to the same right line ED, one of them perpendicular to it, the other not; the perpendicular falls at the side of the acute angle.

For, *if possible*, let BA perpendicular to the line ED fall at the side of the obtuse angle BCE, then the
(1) Def. 12. angle BAE is less than BCE (1); but BAE is greater
(2) Prop. 16. than the same angle BCE (2), which is absurd: BA therefore cannot fall at the side of the obtuse angle, and therefore must fall at the side of the acute angle.

Fig. 33. Cor. 2. Two perpendiculars cannot be drawn from the same point B, to the same right line ED.

For, *if possible*, let the lines BA and BC be both perpendicular to ED, then the angle BAE is equal to
(1) Ax. 11. the angle BCE (1); but it is also greater than it (2),
(2) Prop. 16. which is absurd: therefore the right lines BA and BC cannot both be perpendicular to ED.

PROP. XVII. THEOR.

Fig. 31. *Any two angles of a triangle (BAC) are together less than two right angles.*

Produce any side BC, then the angle ACD is greater
(1) Prop. 16. than either the angle A or B (1); therefore ACB, toge-

ther with either A or B, is less than the same angle ACB together with ACD, that is, less than two right angles (2). In the same manner, if CB be produced from the point B, it can be demonstrated that the angle ABC, together with the angle A, is less than two right angles: therefore any two angles of the triangle are less than two right angles. (2) Prop. 13.

Cor. If any angle of a triangle be obtuse or right, the other two angles are acute: and if two angles be equal to one another, they are acute.

PROP. XVIII. THEOR.

In any triangle (BAC) if one side (AC) be greater than another (AB), the angle opposite to the greater side is greater than the angle which is opposite to the less. Fig. 32.

From the greater side AC cut off the part AD equal to the less (1), and conterminous with it, and join BD. (1) Prop. 3.

The triangle BAD being isosceles (2), the angles ABD and ADB are equal (3); but ADB is greater than the internal angle ACB (4); therefore ABD is greater than ACB, and therefore ABC is greater than ACB: but ABC is opposite the greater side AC, and ACB is opposite the less AB. (2) Constr. (3) Prop. 5. (4) Prop. 16.

PROP. XIX. THEOR.

If in any triangle (BAC) one angle (ABC) be greater than another (C), the side (AC), which is opposite to the greater angle, is greater than the side (AB), which is opposite to the less. Fig. 32.

For the side AC is either equal to, or less or greater than AB.

It is not equal to AB, because the angle ABC would then be equal to ACB (1); which is contrary to the hypothesis. (1) Prop. 5.

It is not less than AB, because the angle ABC would then be less than ACB (2); which is contrary to the hypothesis. (2) Prop. 18.

Since therefore the side AC is neither equal to nor less than AB, it must be greater than it.

Fig. 33. Cor. 1. If from the same point B two right lines BC and BA be drawn to the same right line ED, of which one BC is perpendicular to ED, the other not; BC is less than BA.

For in the triangle ABC the angle BAC is acute, since BCA is right (1); therefore BC is opposite to the less angle, and therefore is less than BA, which is opposite to the greater (2).

(1) Cor. Prop. 17.
(2) Prop. 19.

Fig. 34. Cor. 2. If a line AD be drawn from the vertex of any triangle BAD, to the base BC, it will be less than the greater of the sides containing the angle BAC, if those sides be unequal; or than either of them, if they be equal.

Let the side AB be greater than AC or equal to it, then the angle ACB will be greater than or equal to ABC (1); but the angle ADB is greater than ACD (2), and therefore greater than ABD; therefore the side AB is greater than AD (3). If AC be equal to AB, it is evident it will also be greater than AD.

(1) Prop. 18. & Prop. 5.
(2) Prop. 16.
(3) Prop. 19.

PROP. XX. THEOR.

Fig. 34. See N.

Any two sides (BA and AC) of a triangle (BAC) are together greater than the third side (BC).

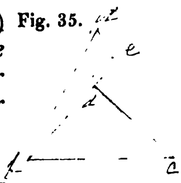
Bisect the angle BAC by the right line AD (1); the external angle BDA is greater than the internal DAC (2); but BAD is equal to DAC (3), therefore BDA is greater than BAD, and therefore the side BA is greater than BD (4): in the same manner it can be demonstrated that the side AC is greater than CD; therefore the two sides BA and AC, taken together, are greater than BD, DC, or the third side BC. Thus by bisecting any angle, it can be demonstrated that the sides containing that angle are greater than the third side.

(1) Prop. 9.
(2) Prop. 16.
(3) Constr.
(4) Prop. 19.

Schol. Hence it is evident, that the difference between two sides of any triangle is less than the third side.

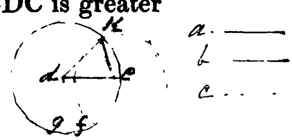
PROP. XXI. THEOR.

Two right lines (DB and DC), drawn to a point (D) within a triangle (BAC) from the extremities of any side (BC), are together less than the sum of the two other sides of the triangle (AB and AC), but contain a greater angle. Fig. 35.



Produce BD to E. The sides BA and AE, of the triangle ABE, are greater than the third side BE (1); add EC to each, and the sides BA and AC are greater than BE and EC; but the sides DE and EC, of the triangle EDC, are greater than the third side DC (1); add BD to each, and BE and EC are greater than BD and DC; but BA and AC are greater than BE and EC, therefore BA and AC are greater than BD and DC. (1)Prop. 20.

Because the external angle BDC is greater than the internal DEC (2), and because for the same reason DEC is greater than BAE; the angle BDC is greater than the angle BAE. (2)Prop. 16.

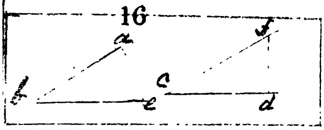


PROP. XXII. PROB.

Given three right lines (A, B and C), of which any two together are greater than the third, to construct a triangle whose sides shall be respectively equal to the given lines. Fig. 36. See N.

From any point D draw the right line DE equal to one of the given lines A (1), and from the same point draw DG equal to another of the given lines B (1). and from the point E draw EF equal to C (1). From the centre D with the radius DG describe a circle, from the centre E with the radius EF describe another circle (2), and from the point of intersection K draw KD and KE. It is evident that the sides DE, DK and KE, of the triangle DKE, are equal to the given right lines A, B and C. (1) Prop. 2. (2) Post. 3.

Cor. In this manner a triangle can be constructed equal to a given one, by constructing a triangle whose sides shall be equal to the sides of the given; for this triangle is equal to the given one (1). (1) Schol. Prop. 8.



PROP. XXIII. PROB.

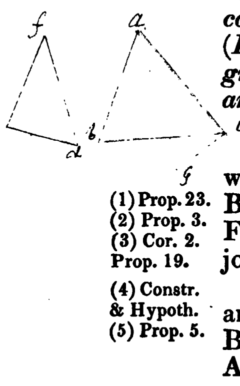
Fig. 37. *At a given point (B), in a given right line (BE), to make an angle equal to a given angle (C).*

In the sides of the given angle take any points D and F; join DF; and construct a triangle EBA, which shall be equilateral with the triangle DCF, and whose sides AB and EB meeting at the given point B shall be
(1) Prop. 22. equal to FC and DC, the legs of the given angle C (1). The angle EBA is equal to the given angle DCF.

For, as the triangles DCF and EBA have all their sides respectively equal, the angles FCD and ABE, op-
(2) Prop. 8. posite the equal sides DF and EA, are equal (2).

PROP. XXIV. THEOR.

Fig. 38. See N. *If two triangles (EFD, BAC) have two sides of the one respectively equal to two sides of the other (FE to AB, and FD to AC), and if one of the angles (BAC) contained by the equal sides be greater than the other (EFD), the side (BC) opposite to that greater angle is greater than the side (ED), which is opposite to the less angle.*



From the point A draw the right line AG, making with the side AB, which is not the greater, an angle
(1) Prop. 23. BAG equal to the angle EFD (1); make AG equal to
(2) Prop. 3. FD (2); and, as the point G must fall below the base (3),
(3) Cor. 2. Prop. 19. join BG, GC.

(4) Constr. & Hypoth. Since the right lines AG and AC are equal (4), the angles ACG and AGC are equal (5); but the angle
(5) Prop. 5. BGC is greater than AGC, therefore greater than ACG, and therefore greater than BCG. Then, in the triangle BGC, the angle BGC is greater than BCG,
(6) Prop. 19. therefore the side BC is greater than BG (6); but BG
(7) Constr. & Prop. 4. is equal to ED (7), and therefore BC is greater than ED.

PROP. XXV. THEOR.

If two triangles (BAC and EFD) have two sides of the one respectively equal to two sides of the other (BA to EF, and AC to FD), and if the third side (BC) of the one be greater than the third side (ED) of the other, the angle (A) opposite to the greater side is greater than the angle (F), which is opposite to the less. Fig. 39.

The angle A is either equal to the angle F, or less than it, or greater than it.

It is not equal; for, if it were, the side BC would be equal to the side ED (1): which is contrary to the hypothesis. (1) Prop. 4.

It is not less; for, if it were, the side BC would be less than the side ED (2): which is contrary to the hypothesis. (2) Prop. 24.

Since therefore the angle A is neither equal to, nor less than F, it is greater.

PROP. XXVI. THEOR.

If two triangles (BAC, DEF) have two angles of the one respectively equal to two angles of the other (B to D, and C to F), and a side of the one equal to a side of the other, (BC to DF or BA to DE), either the sides adjacent to or opposite to those equal angles; the remaining sides and angles are respectively equal to one another. Fig. 40. See N.

First, let the equal sides be BC and DF, which are adjacent to the equal angles; then the side BA is equal to the side DE.

For, *if it be possible*, let one of them BA be greater than the other; make BG equal to DE, and join CG.

In the triangles GBC, EDF, the sides GB, BC are respectively equal to the sides ED, DF (1), and the angle B is equal to the angle D (2), therefore the angles BCG and DFE are equal (3); but the angle BCA (1) Constr. & Hypoth. (2) Hypoth. (3) Prop. 4.

(4) Hypoth. is also equal to DFE (4), therefore the angle BCG is
(5) Ax. 1. equal to BCA (5), which is absurd: neither of the sides BA and DE therefore is greater than the other, therefore they are equal; and also BC and DF are equal (2), and the angles B and D are equal (2); therefore the side AC is equal to the side EF, as also the
(6) Prop. 4. angle A to the angle E (6).

Next, let the equal sides be BA and DE, which are opposite to the equal angles C and F; and the sides BC and DF shall also be equal.

For, *if it be possible*, let one of them BC be greater than the other; make BH equal to DF, and join AH.

In the triangles ABH, EDF, the sides AB, BH are
(1) Constr. & Hypoth. respectively equal to the sides ED, DF (1), and the an-
(2) Hypoth. gle B is equal to the angle D (2), therefore the angles
(3) Prop. 4. AHB and EFD are equal (3); but the angle C is also
(4) Ax. 1. equal to EFD (2), therefore AHB and C are equal (4),
(5) Prop. 16. which is impossible (5): neither, therefore, of the sides BC and DF is greater than the other, therefore they are equal; but BA and DE are also equal, as also the angles B and D (2); therefore the side AC is equal to the side EF, and also the angle A to the angle E (3).

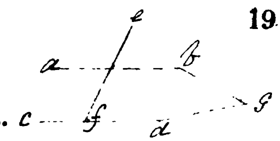
Schol. It is evident that the triangles themselves are equal.

Fig. 41. Cor. 1. In an isosceles triangle ABC the right line BD, drawn from the vertex perpendicular to the base, bisects the base and the vertical angle.

For in the triangles, ABD, CBD, the angles A and ADB are respectively equal to the angles C and CDB
(1) Hypoth. (1), and the side BD, which is opposite to the equal angles A and C, is common to both triangles; therefore the angles ABD and CBD are equal, and also the sides
(2) Prop. 26. AD and DC (2), and therefore the base and vertical angle are bisected.

Cor. 2. It is evident that the right line, which bisects the vertical angle of an isosceles triangle, bisects the base and is perpendicular to it; and that the right line drawn from the vertical angle, bisecting the base, is perpendicular to it, and bisects the vertical angle.

PROP. XXVII. THEOR.



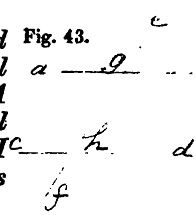
If a right line (EF) intersect two right lines (AB and CD), and make alternate angles equal to each other (AEF to EFD, or BEF to EFC), these right lines are parallel. Fig. 42. See N.

For, *if it be possible*, let these lines not be parallel, but meet in G; the external angle AEF of the triangle EGF is greater than the internal EFG (1); but it is also equal to it (2), which is absurd: therefore the lines AB and CD do not meet at the side BD; and in the same manner it can be demonstrated, that they do not meet at the side AC. Since, then, the right lines do not meet on either side, they are parallel (3).

(1) Prop. 16. (2) Hypoth. (3) Def. 28.

PROP. XXVIII. THEOR.

If a right line (EF) intersect two right lines (AB and CD), and make an external angle equal to the internal and opposite angle upon the same side of the line (EGA to GHC, or EGB to GHD), or make internal angles at the same side (AGH and CHG, or BGH and DHG) equal to two right angles, the two right lines are parallel to one another. Fig. 43.



First, let the angles EGA and GHC be equal; and since the angle EGA is equal to BGH (1), the angles GHC and BGH are equal (2); but they are alternate angles, therefore the right lines AB and CD are parallel (3).

(1) Prop. 15. (2) Ax. 1. (3) Prop. 27.

In the same manner the proposition can be demonstrated, if the angles EGB and GHD be given equal.

Next, let the angles AGH and CHG taken together be equal to two right angles; since the angles GHD and CHG, taken together, are also equal to two right angles, (4), the angles AGH and CHG taken together are equal to the angles GHD and CHG taken together (2); take away the common angle CHG, and the remaining angles AGH and GHD are equal; but they are

(4) Prop. 13.

alternate angles, and therefore the right lines AB
(3) Prop.27. and CD are parallel (3). In the same manner the proposition can be demonstrated, if the angles BGH and DHG were given equal to two right angles.

PROP. XXIX. THEOR.

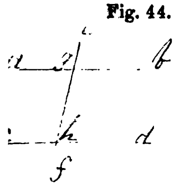


Fig. 44. *If a right line (EF) intersect two parallel right lines (AB and CD), it makes the alternate angles equal (AGH to GHD, and CHG to HGB): and the external angle equal to the internal and opposite upon the same side (EGA to GHC, and EGB to GHD); and also the two internal angles at the same side (AGH and CHG, BGH and DHG) together equal to two right angles.*

First. The alternate angles AGH and GHD are equal. For, *if it be possible*, let one of them AGH be greater than the other, and, by adding the angle BGH to both, AGH and BGH together are greater than BGH and GHD together; but AGH and BGH together are
(1) Prop. 13. equal to two right angles (1); therefore BGH and GHD together are less than two right angles; and therefore the lines AB and CD, if produced, would
(2) Ax. 12. meet at the side BD (2); but they are parallel (3), and
(3) Hypoth. therefore cannot meet, which is absurd: therefore neither of the angles AGH and GHD is greater than the other; they are therefore equal. In the same manner it can be demonstrated that the angles BGH and GHC are equal.

Secondly. The external angle EGB is equal to the internal GHD. For the angle EGB is equal to the
(4) Prop. 15. angle AGH (4), and AGH is equal to the alternate
(5) Part. pr. angle GHD (5), therefore EGB is equal to GHD (6).
(6) Ax. 1. In the same manner it can be demonstrated that EGA and GHC are equal.

Thirdly. The internal angles at the same side, BGH and GHD, together are equal to two right angles. For, since the alternate angles GHD and AGH are equal (5), if the angle BGH be added to both, BGH and GHD together are equal to BGH and AGH together, and therefore are equal to two right angles (1). In the same manner it can be demonstrated that the angles AGH and GHC together are equal to two right angles.

PROP. XXX. THEOR.

If two right lines (AB, CF) be parallel to the same right line (DN), they are parallel to each other. Fig. 45. See N.

Let the right line GP intersect them, the external angle GLB is equal to the internal LON (1), and also the angle LON is equal to OPF (1); therefore GLB is equal to OPF (2), and therefore the right lines AB and CF are parallel (3). (1) Prop. 29. (2) Ax. 1. (3) Prop. 28.

PROP. XXXI. PROB.

Through a given point (C), to draw a right line parallel to a given right line (AB). Fig. 46.

In the line AB take any point F; join CF, and at the point C and with the right line CF make the angle FCE equal to AFC (1), but at the opposite side of the line CF: the line DE is parallel to AB. (1) Prop. 23.

For the right line FC, intersecting the lines DE and AB, makes the alternate angles ECF and AFC equal, and therefore the lines are parallel (2). (2) Prop. 27.

PROP. XXXII. THEOR.

If any side (AB) of a triangle (ABC) be produced, the external angle (FBC) is equal to the sum of the two internal and opposite angles (A and C): and the three internal angles of every triangle, taken together, are equal to two right angles. Fig. 47.

Through B draw BE parallel to AC (1). (1) Prop. 31.

The angle FBE is equal to the internal angle A (2), and the angle EBC is equal to the alternate C (2); therefore the whole external angle FBC is equal to the two internal angles A and C. (2) Prop. 29.

The angle ABC with FBC is equal to two right angles (3); but FBC is equal to the two angles A and C (3) Prop. 13.

(4) Part Pr. (4), therefore the angle ABC, together with the angles A and C, is equal to two right angles.

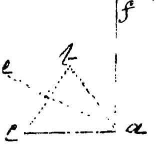
Cor. 1. Hence, in a triangle, if one angle be right, the other two are together equal to a right angle; and if one angle be equal to the other two, it is a right angle.

Cor. 2. If two triangles have two angles in the one respectively equal to two angles in the other, the remaining angles are also equal.

Cor. 3. In a right-angled isosceles triangle, each angle at the base is half a right angle.

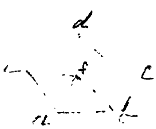
Cor. 4. Each angle of an equilateral triangle is equal to a third of two right angles.

Fig. 48. See N.



Cor. 5. Hence can be derived a method of trisecting a right angle FAC: take any portion of the side AC, and construct upon it an equilateral triangle CBA, and bisect the angle CAB by the right line AE: since CAB is equal to a third part of two right angles, it must be equal to two-thirds of one right angle; therefore BAF is the third part of a right angle, and therefore equal to BAE and EAC.

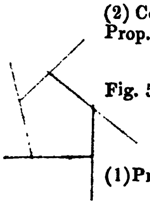
Fig. 49. See N.



Cor. 6. All the internal angles of any rectilineal figure, ABCDE, together with four right angles, are equal to twice as many right angles as the figure has sides.

Take any point F within the figure, and draw the right lines FA, FB, FC, FD, and FE. There are formed as many triangles as the figure has sides, and therefore all their angles taken together are equal to
(1) Prop. 32. twice as many right angles as the figure has sides (1): but the angles at the point F are equal to four right
(2) Cor. 3. Prop. 13. angles (2); and therefore the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

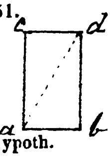
Fig. 50.



Cor. 7. The external angles of any rectilineal figure are together equal to four right angles. For each external angle, with the internal adjacent to it, is equal to
(1) Prop. 13. two right angles (1); therefore all the external angles, with all the internal, are equal to twice as many right angles as the figure has sides: but the internal angles, together with four right angles, are equal to twice as
(2) Cor. pr. many right angles as the figure has sides (2); take away from both the internal angles, and the external angles
(3) Ax. 3. remain equal to four right angles (3).

PROP. XXXIII. THEOR.

Right lines (AC and BD), which join the adjacent extremities of two equal and parallel right lines (AB and CD), are themselves equal and parallel. Fig. 51.

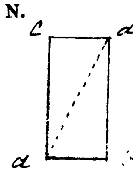


Draw the diagonal AD; and in the triangles CDA and BAD, the sides CD and BA are equal (1), AD is common to both triangles, and the angle CDA is equal to the alternate BAD (2); therefore the lines AC and BD are equal (3), and also the angles CAD and BDA (3); therefore the right line AD, cutting the right lines AC and BD, makes alternate angles equal, and therefore the right lines AC and BD are parallel (4).

(1) Hypoth.
(2) Prop. 29.
(3) Prop. 4.
(4) Prop. 27.

PROP. XXXIV. THEOR.

The opposite sides (AB and CD, AC and BD) of a parallelogram (AD) are equal to one another, as are also the opposite angles (A and D, C and B), and the parallelogram itself is bisected by its diagonal (AD). Fig. 51. See N.



For in the triangles CDA, BAD, the alternate angles CDA and BAD, CAD and BDA are equal to one another (1), and the side AD between the equal angles is common to both triangles, therefore the sides CD and CA are equal to AB and BD (2), and the triangle CDA is equal to the triangle BAD (3), and the angles ACD and ABD are also equal (2); and since the angle ACD with CAB is equal to two right angles (1), and ABD with CDB is equal to two right angles, take away from both the equals ACD and ABD, and the remaining angles CAB and CDB are equal (4).

(1) Prop. 29.
(2) Prop. 26.
(3) Schol. Prop. 26.
(4) Ax. 3.

Cor. 1. If one angle of a parallelogram be a right angle, the other three are right angles. For the adjacent angle is right, because with a right angle it is equal to two right angles (1); and the opposite are right angles, because they are equal to these right angles (2).

(1) Prop. 29.
(2) Prop. 34.

Cor. 2. If two parallelograms have one angle equal in each, the other angles are respectively equal. For

the angles opposite these equal angles are equal to
(1) Prop. 34. them (1), and therefore to each other (2), but the
(2) Ax. 1. angles adjacent to them are also equal to each other, as being, together with these equals, equal to two right
(3) Prop. 29. angles (3).

PROP. XXXV. THEOR.

Fig. 52, 53, 54. See N. *Parallelograms (BD and BF), on the same base and between the same parallels, are equal.*

Produce the side BC to G.

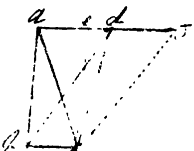
Because the lines AB and DC are parallel, the angles
(1) Prop. 29. ABG and DCG are equal (1), and because EB and FC are parallel, the angles EBG and FCG are also equal
(2) Ax. 3. (1); therefore the angles ABE and DCF are equal (2),
(3) Prop. 34. and the sides AB and BE are equal to DC and CF (3);
(4) Prop. 4. therefore the triangles ABE and DCF are equal (4): take away the triangle ABE from the quadrilateral ABCF, the remainder is the parallelogram BF, and from the same quadrilateral take away the triangle DCF, the remainder is the parallelogram BD, therefore the parallelograms BF and BD are equal (2).

PROP. XXXVI. THEOR.

Fig. 55. *Parallelograms (BD and EG), on equal bases and between the same parallels, are equal.*

Draw the right lines BF and CG.

Because the lines BC and FG are equal to the same EH
(1) Hypoth. & Prop. 34. (1), they are equal to one another (2); but they are also
(2) Ax. 1. parallel (3); therefore BF and CG which join their ex-
(3) Hypoth. tremities are parallel (4), and BG is a parallelogram,
(4) Prop. 33. therefore equal to both BD and EG (5); and therefore
(5) Prop. 35. the parallelograms BD and EG are equal (2).

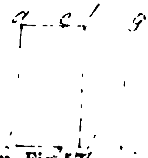


PROP. XXXVII. THEOR.

Triangles (BAC and BDC) on the same base, and between the same parallels, are equal. Fig. 56.

Draw through the point C the right lines CE and CF parallel to BA and BD (1): the parallelograms BAEC, BDFC are equal (2); but the triangles BAC and BDC are halves of them (3), and therefore equal (4).

(1) Prop. 31.
(2) Prop. 35.
(3) Prop. 34.
(4) Ax. 7.



PROP. XXXVIII. THEOR.

Triangles (BAC and HDE) on equal bases, and between the same parallels, are equal. Fig. 57.

Draw, through the points C and E, the right lines CF and EG parallel to BA and HD (1): the parallelograms BAFC, HDGE are equal (2); but the triangles BAC and HDE are halves of them (3), and therefore equal (4).

(1) Prop. 31.
(2) Prop. 36.
(3) Prop. 34.
(4) Ax. 7.

PROP. XXXIX. THEOR.

Equal triangles (BAC and BDC), on the same base and on the same side of it, are between the same parallels. Fig. 58.

If the right line AD, which joins the vertices of the triangles, be not parallel to BC, draw through the point A a right line AF parallel to BC, cutting a side BD of the triangle BDC, or the side produced, in a point E different from the vertex; and draw CE.

Because the right lines AF and BC are parallel, the triangle BEC is equal to BAC (1); but BDC is also equal to BAC (2); therefore BEC and BDC are equal (3), a part equal to the whole, which is absurd. There-

(1) Prop. 37.
(2) Hypoth.
(3) Ax. 1.

fore the line AF is not parallel to BC: and in the same manner it can be demonstrated that no other line, except AD, is parallel to it; therefore AD is parallel to BC.

PROP. XL. THEOR.

Fig. 59. See N.

Equal triangles (BAC and GDH), on equal bases and on the same side, are between the same parallels.

If the right line AD, which joins the vertices of the two triangles, be not parallel to BH, draw through the point A the right line AF parallel to BH, cutting a side GD of the triangle GDH, or the side produced, in a point E different from the vertex; and join HE.

Because the right line AF is parallel to BH, and BC and GH are equal, the triangle GEH is equal to BAC (1); but GDH is also equal to BAC (2); therefore GEH and GDH are equal (3), a part equal to the whole, which is absurd. Therefore AF is not parallel to BH: and in the same manner it can be demonstrated that no other line, except AD, is parallel to BH; therefore AD is parallel to BH.

(1) Prop. 38. (2) Hypoth. (3) Ax. 1.

PROP. XLI. THEOR.

Fig. 60.

If a parallelogram (BF) and a triangle (BAC) have the same base, and be between the same parallels, the parallelogram is double of the triangle.

Draw CD. The triangle BDC is equal to the triangle BAC (1); but BF is double of the triangle BDC (2); therefore BF is also double of the triangle BAC.

(1) Prop. 37. (2) Prop. 34.

Schol. Hence it is evident that a parallelogram is double of a triangle, if they have equal bases, and be between the same parallels.

Fig. 61.

Cor. If a triangle BAC and a parallelogram DF be between the same parallels, and the base BC of the triangle be double the base DC of the parallelogram, the triangle is equal to the parallelogram.

Draw AD. Because the triangles BAD and DAC are equal (1); BAC is double of the triangle DAC, but (1) Prop. 38. the parallelogram DF is double of the same DAC (2); (2) Prop. 41. therefore the triangle BAC and the parallelogram DF (3) Ax. 6. are equal (3).

PROP. XLII. PROB.

To construct a parallelogram equal to a given triangle (BAC), and having an angle equal to a given one (O). Fig. 61.

Through the point A draw the right line AF parallel to BC; bisect BC the base of the triangle in D; and at the point D, and with the right line CD, make the angle CDE equal to the given one O; through C draw CF parallel to DE, until it meet the line AF in F: DF is the required parallelogram.

Because EF is parallel to DC (1), and CF parallel to (1) Constr. DE (1), DEFC is a parallelogram (2): it has also the (2) Def. 30. angle CDE equal to the given one O (1): and it is equal to the triangle BAC, because it is between the (3) Cor. same parallels and on half the base of the triangle (3). Prop. 41.

PROP. XLIII. THEOR.

In a parallelogram (AC) the complements (AK and KC), of the parallelograms (FH and GE) about the diagonal, are equal. Fig. 62.

Draw the diagonal DB, and, through any point in it K, draw the right lines FE and GH parallel to AB and BC: then FH and GE are the parallelograms about the diagonal, AK and KC their complements.

Because the triangles BAD and BCD are equal (1), (1) Prop. 34. and the triangles BGK, KFD are equal to BEK, KHD (1), take away the equals BGK and KFD, BEK and KHD, from the equals BAD and BCD, and the remainders, namely the complements AK and KC, are equal (2). (2) Ax. 3.

Cor. The parallelograms OF and EK about the diagonal of a square AD are squares. Fig. 63.

Because the triangle BAC is isosceles, and the angle

(1) Cor. 3. Prop. 32. at A right, ABC is half a right angle (1): since, then, in the triangle OBG, the angle at O is right, for it is (2) Prop. 29. external to a right angle (2), and the angle OBG half a right angle, the angle OGB must also be half a right (3) Prop. 32. angle (3); and therefore the sides OG and OB are equal (4) Prop. 6. (4); it is evident therefore, that OF is a square. In the same manner it can be demonstrated that EK is a square.

PROP. XLIV. PROB.

Fig. 64. See N. *To a given right line (OS) to apply a parallelogram, which shall be equal to a given triangle, and shall have an angle equal to a given one (V).*

Let the given triangle be GHO, one of whose sides GO and the given line OS form one right line.

Bisect GO in R; upon RO construct a parallelogram RC equal to the given triangle, and having an angle (1) Prop. 42. BRO equal to the given one V (1); through S draw SD parallel to OC or RB, until it meet BC produced to D; draw DO to meet BR produced to A; through A draw AL parallel to RS or to BD; and produce CO and DS to F and L. The parallelogram FS is applied to the given line OS, is equal to the given triangle GHO, and has the angle OFL equal to the given V.

(2) Constr. Because ABDL is a parallelogram (2), and in it FS and RC are the complements of the parallelograms (3) Prop. 43. about the diagonal, FS is equal to RC (3), but RC is equal to the given triangle GHO (2); therefore FS is (4) Ax. 1. equal to GHO (4); and because the angle OFL is (5) Prop. 29. equal to the internal one BAF (5), and BAF equal to the external one BRO (5), OFL is equal to BRO; but BRO is equal to the given V (2), therefore OFL is equal to V.

If the given triangle be KNO, which has not a side forming one right line with the given, produce the given line OS, until the produced part be equal to any side KO of the given triangle; and upon it construct a tri- (6) Cor. Prop. 22. angle GHO equal to KNO (6): apply to the given line (7) Part. pr. OS a parallelogram equal to GHO (7), and it shall be equal to KNO; as is evident.

PROP. XLV. PROB.

To construct a parallelogram equal to a given rectilineal figure (ABCED), and having an angle equal to a given one. Fig. 65. See N.

Resolve the given rectilineal figure into triangles.

Construct a parallelogram RQ equal to the triangle BDA, and having an angle RIQ equal to the given H (1); on a side of it RV construct the parallelogram XV (1) Prop. 42. equal to the triangle CBD, and having an angle equal to the given one (2); and so on construct parallelograms (2) Prop. 44. equal to all the triangles, into which the figure is resolved: ILYQ is a parallelogram, equal to the given rectilineal figure, and having an angle LIQ equal to the given one H.

Because RV and IQ are parallel, the angle VRI, together with QIR, is equal to two right angles (3); but (3) Prop. 29. VRX is equal to QIR (4); therefore VRI with VRX (4) Constr. is equal to two right angles (5), and therefore IR and (5) Ax. 2. RX form one right line (6); in the same manner it can (6) Prop. 14. be demonstrated that RX and XL form one right line, therefore IL is a right line; and because QV is parallel to IR, the angle QVR, together with VRI, is equal to two right angles (3); but IR is parallel to VF, and therefore IRV is equal to FVR (3), and therefore QVR with FVR is equal to two right angles, and QV and FV form one right line (6); in the same manner it can be demonstrated that VF and FY form one right line; therefore QY is a right line, and also it is parallel to IL (4): and because LY and RV are parallel to the same line XF, LY is parallel to RV (7); but (7) Prop. 30. IQ and RV are parallel, therefore LY is parallel to IQ (7), and therefore LIQY is a parallelogram (8): and (8) Def. 30. it has the angle LIQ equal to the given H, and is equal (9) Constr. & Ax. 2. to the given rectilineal figure ABCED (9).

Cor. 1. Hence a parallelogram can be applied to a given right line and in a given angle, equal to a given rectilineal figure, by applying to the given line a parallelogram equal to the first triangle.

Cor. 2. In the same manner a parallelogram can be applied to a given line, equal to the sum of two or more rectilineal figures.

Cor. 3. A parallelogram can be constructed equal to the difference between two given rectilineal figures, by applying to the same right line at the same side of it, and in the same angle, parallelograms equal to the two figures: the difference between these parallelograms is a parallelogram, and equal to the difference of the given figures.

PROP. XLVI. PROB.

Fig. 66. See N. *On a given right line (AB) to describe a square.*

From either extremity of the given line AB draw a
(1) Schol. Prop. 11. line AC perpendicular (1) and equal to it (2), through
(2) Prop. 3. C draw CD parallel to AB (3), and through B draw
(3) Prop. 31. BD parallel to AC (3); ABDC is the required square.
(4) Constr. Because ABDC is a parallelogram (4), and the angle A a right angle (4), the angles C, D and B are also right
(5) Cor. 1. Prop. 34. (5); and because AC is equal to AB (4), and the sides CD and DB are equal to AB and AC (6), the four sides
(6) Prop. 34. AB, AC, CD, DB, are equal; therefore ABDC is a
(7) Def. 31. square (7).

Fig. 67. Cor. 1. The squares of equal right lines AD and XS are equal.

Draw the diagonals BD and YS. Because the right
(1) Hypoth. & Def. 31. lines BA, AD, are equal to XY, XS (1), and the an-
(2) Prop. 4. gles A and X equal, the triangles BAD, YXS are
(3) Prop. 34. equal (2), and therefore the squares AC and XZ, which
(4) Ax. 6. are double of the triangles (3), are equal (4).

Fig. 67. Cor. 2. If two squares AC and XZ be equal, their sides are equal.

For, *if it be possible*, let one of them AD be the greater;
(1) Prop. 3. take the line AF equal to XS (1), and AE equal to XY, and join EF. The triangle EAF is equal to YXS
(2) Prop. 4. (2); but YXS is equal to BAD, as they are halves of the equal squares AC and XZ; therefore EAF is equal
(3) Ax. 1. to BAD (3), a part to the whole, which is absurd: therefore neither AD nor XS is greater than the other: therefore they are equal.

In a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the sides containing the angle.

PROP. XLVII. THEOR.

In a right-angled triangle (ABC) the square which is described upon the side (AC) subtending the right angle, is equal to the sum of the squares described upon the sides (AB and CB) which contain the right angle. Fig. 68. See N.

On the sides AB, AC and BC, describe the squares AX, AF and BI; draw BE parallel to either CF or AD; and join BF and AI.

Because the angles ICB and ACF are equal (1), if BCA be added to both, the angles ICA and BCF are equal; and the sides IC, CA, are equal to the sides BC, CF (2), therefore the triangles ICA and BCF are equal (3); but AZ is parallel to CI (4), therefore the parallelogram CZ is double of the triangle ICA, as they are upon the same base CI, and between the same parallels (5); and the parallelogram CE is double of the triangle BCF, as they are upon the same base CF and between the same parallels (5); therefore the parallelograms CZ and CE, being double of the equal triangles ICA and BCF, are equal to one another (6): in the same manner it can be demonstrated that AX and AE are equal; therefore the whole DACF is equal to the sum of CZ and AX.

(1) Ax. 11. (2) Def. 31. (3) Prop. 4. (4) Prop. 14. (5) Prop. 41. (6) Ax. 6.

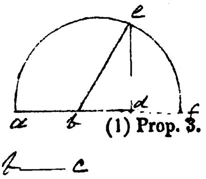
Cor. 1. If any two sides of a right-angled triangle be given in numbers, the third side can be found; for it is the square root of the sum or difference of the squares of the given lines, according as the given sides contain the right angle or not.

Cor. 2. To find a square equal to the sum of two or more given squares. Fig. 69.

Let the right lines A, B and C, be sides of the given squares: construct a right angle FDE, and take on the sides of it the lines DG and DH equal to the given lines A and B (1); draw GH, and take DI and DK equal to GH and to the given line C: the square of IK is equal to the squares of A, B and C, for it is equal to the squares of DI and DK (2); but the square of DI is equal to the squares of DG and DH (2); therefore the square of IK is equal to the squares of DK, DH and DG, or of the lines C, B and A, which are equal to them.

(1) Prop. 3. (2) Prop. 47.

Cor. 3. To find a line, whose square is equal to the Fig. 70.



difference between the squares of two given lines AB and BC.

(1) Prop. 3. Produce either of the given lines AB, till the part produced is equal to the other (1); from B as a centre, and with the radius AB, describe a semicircle AEF; through D draw DE perpendicular to AB, and it is the required line.

(2) Prop. 47. For draw BE. The square of BE, or of BA which is equal to BE, is equal to the squares of BD and DE (2); therefore, if the square of BD be taken away from the square of BA, the remainder is equal to the square of DE.

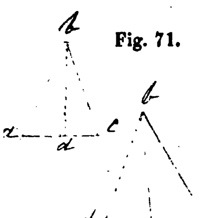


Fig. 71. Cor. 4. If from any angle of a triangle ABC, a perpendicular be drawn to the opposite side, the difference between the squares of the sides AB and BC, which contain that angle, is equal to the difference between the squares of the segments AD and DC of the side, on which the perpendicular falls.

(1) Prop. 47. (2) Ax. 3. For the square of the side AB is equal to the squares of AD and DB (1), and the square of BC is equal to the squares of BD and DC (1); therefore the difference between the squares of AB and BC is equal to the difference between the sum of the squares of AD and DB and the sum of the squares of CD and DB (2); or, taking away the common square of DB, equal to the difference between the squares of AD and CD.

Fig. 71. Cor. 5. The excess of the square of either side above the square of the conterminous segment of the third side is the same. For it is evident that the excess is

(1) Prop. 47. the square of the perpendicular BD (1).

PROP. XLVIII. THEOR.

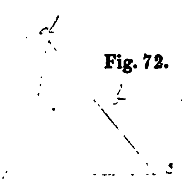


Fig. 72. *If the square described upon one side (AC) of a triangle (ABC) be equal to the sum of the squares described upon the other two sides (AB and BC), the angle (ABC) opposite to that side is a right angle.*

(1) Schol. Prop. 11. (2) Prop. 3. (3) Prop. 47. From the point B draw BD, perpendicular (1) to one of the sides AB, and equal (2) to the other BC; and join AD. The square of AD is equal to the squares of AB and BD (3), or to the squares of AB and of BC which is

equal to BD (4): but the squares of AB and BC are (4) Constr.
equal to the squares of AC (5); therefore the squares (5) Hypoth.
of AD and AC are equal, and therefore the lines themselves are equal (6): but also DB and BC are equal, (6) Cor. 2. Prop. 46.
and the side AB is common to both triangles; therefore the triangles ABC and ABD are mutually equilateral, and therefore the angle ABC is equal to the angle ABD (7); but ABD is a right angle (4), therefore ABC (7) Prop. 8.
is also a right angle.

THE

ELEMENTS OF EUCLID.

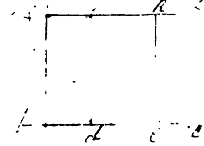
BOOK II.

DEFINITIONS.

1. Every rectangle or right-angled parallelogram is said to be contained by the two right lines, which contain one of its right angles.

(It is called the rectangle under these two lines.)

Plate 2. Fig. 6. 2. In any parallelogram either of the parallelograms (EK or OF) about the diagonal, together with the two complements (AG and GD) is called a Gnomon.



PROP. I. THEOR.

If there be two right lines (A and BC), one of which is divided into any number of parts (BD, DE, EC), the rectangle under the two lines is equal to the sum of the rectangles under the undivided line (A) and the several parts of the divided line (BD, DE, EC). Plate 2. Fig. 1.

From the point B draw BH perpendicular to BC; take on it BF equal to A; and through F draw FL parallel to BC, and draw DG, EK, and CL, parallel to BF.

It is evident that the rectangle BL is equal to the rectangles BG, DK and EL; but the rectangle BL is the rectangle under A and BC, for BF is equal to A: and the rectangles BG, DK, EL, are the rectangles under A and BD, A and DE, and A and EC; for each of the lines, BF, DG and EK, is equal to A (1). (1) Constr. & Prop. 34. B. 1.

Cor. Hence, and from prop. 34. B. 1. it appears that the area of a rectangle is found, by multiplying its altitude into the base: and from prop. 35. and 36. B. 1. it also appears that the area of any parallelogram is found, by multiplying its altitude into the base: and from prop. 37. and 38. B. I. that the area of a triangle is found by multiplying its altitude into half the base.

PROP. II. THEOR.

If a right line (AB) be divided into any two parts (in C), the square of the whole line is equal to the sum of the rectangles under the whole (AB) and each of the parts (AC, CB). Fig. 2. See N.

On AB describe the square ADFB (1); and through C draw CE parallel to AD. The square AF is equal to the rectangles AE and CF. But the rectangle AE is the rectangle under AB and AC, because AD is equal (1) Prop. 46. B. 1.

(1) Constr. (3) Prop. 34. B. 1. to AB (2), and the rectangle CF is the rectangle under AB and CB, because CE is equal to AB (3).

Otherwise thus:

Fig. 3. Assume a right line X, equal to the given line AB. The rectangle under X and AB, or the square of AB (1), is equal to the sum of the rectangles under X and AC and under X and CB (2); that is, to the sum of the rectangles under AB and AC, and under AB and CB.

(1) Hypoth. (2) Prop. 1.

PROP. III. THEOR.

Fig. 4. *If a right line (AB) be divided into any two parts (in C), the rectangle under the whole line (AB) and either part (AC) is equal to the square of that part (AC) together with the rectangle under the parts (AC and CB).*

On AC describe the square ADFC; and through B draw BE parallel to AD, until it meet DF produced to E. The rectangle AE is equal to the square ADFC together with the rectangle CE.

But the rectangle AE is the rectangle under AC and AB, for AD is equal to AC (1), and the square ADFC is the square of AC (2), and the rectangle CE is the rectangle under AC and CB, for CF is equal to AC (1).

(1) Const. & Def. 31. B. 1. (2) Constr.

Otherwise thus:

Fig. 5. Assume a right line X equal to AC.

The rectangle under X and AB is equal to the sum of the rectangles under X and AC, and under X and CB (1): but the rectangle under X and AB is the rectangle under AC and AB; and the rectangle under X and AC is the square of AC, and the rectangle under X and CB is the rectangle under AC and CB.

(1) Prop. 1.

PROP. IV. THEOR.

If a right line (AB) be divided into any two parts (in O), the square of the whole line is equal to the sum of the squares of the parts and twice the rectangle under the parts. Fig. 6.

On AB describe the square ACDB; draw CB; and through O draw OK parallel to AC, cutting CB in G; and through G draw EF parallel to AB.

The square ACDB is equal to the squares EK and OF, together with the rectangles AG and GD. (1)Constr.& Cor. Prop. 43. B.1.

But OF is the square of OB (1), and EK is the square of AO, for EG is equal to AO (2); and AG and GD together are equal to double the rectangle under the parts, because GD is equal to AG (3); and AG is the rectangle under the parts AO and OB, because OG and OB are equal (4). (2)Constr.& Prop34.B.1. (3) Prop.43. B.1. (4) Def. 31. B.1.

Otherwise thus:

The square of AB is equal to the sum of the rectangles under AB and AO, and under AB and BO (1): but the rectangle under AB and AO is equal to the sum of the rectangle under AO and OB, and the square of AO (2); and the rectangle under AB and BO is equal to the sum of the rectangle under AO and OB and the square of OB (2), therefore the sum of the rectangles under AB and AO, and under AB and BO, or the square of AB (1), is equal to the sum of the squares of AO and OB and double the rectangle under AO and OB. Fig. 7. (1) Prop. 2. (2) Prop. 3.

Cor. Hence it is evident that the square of half the line is the fourth part of the square of the whole line: for, when the line is bisected, the rectangle under the parts is equal to the square of half the line.

PROP. V. THEOR.

Fig. 8. *If a right line (AB) be cut into equal parts (in C), and into unequal (in D), the rectangle under the unequal parts (AD and DB), together with the square of the intermediate part (CD), is equal to the square of half the line (CB).*

On CB describe the square CKMB; draw KB; and through the point D draw DL parallel to CK, and cutting KB in G, and through G draw HGE parallel to AB, until it meet the line AE drawn through A parallel to CK.

(1) Hypoth. (2) Prop. 36. B. 1. (3) Prop. 43. B. 1. (4) Ax. 2. (5) Cor. Prop. 17. & Def. 15. B. 1. (6) Prop. 34. B. 1.

Because the lines AC and CB are equal (1), the rectangles AF and CH are equal (2); but the rectangles CG and GM are also equal (3), therefore the rectangle AG is equal to the gnomon CHL (4): add to both the square FL, and the rectangle AG, together with the square FL, is equal to the square CKMB. But the rectangle AG is the rectangle under AD and DB, for DG is equal to DB (5); and FL is the square of CD, because FG and CD are equal (6); and CKMB is the square of CB.

Otherwise thus:

Fig. 9. (1) prop. 1. (2) Prop. 3. (3) Prop. 4.

The rectangle under AD and DB is equal to the sum of the rectangles under AC and DB and under CD and DB (1); but the rectangle under AC and DB is equal to the rectangle under CB and DB, because AC and CB are equal, or to the rectangle under CD and DB, together with the square of DB (2): add to both the square of CD; and the rectangle under AD and DB, together with the square of CD, is equal to double the rectangle under CD and DB, together with the squares of CD and DB; that is, to the square of CB (3).

(1) Prop. 1.

Cor. 1. Hence it is evident that, if any line be bisected, the rectangle under the parts is greater than if it be cut unequally, and therefore the sum of the squares of the parts is less (1).

Cor. 2. If two equal right lines be so divided, that the rectangle under the segments of one be equal to the rectangle under the segments of the other, the segments are equal.

If one of the right lines be bisected, then the other is also bisected; for the rectangle under the parts is then equal to the square of the half line, whence it is evident that the segments themselves are equal.

But if they be not bisected, let AB and CD be the given equal lines, and let them be divided in E and F. Let the lines be bisected in G and H; and, as the lines are equal (1), their halves are equal, and therefore the squares of the halves (2); but the rectangles under AE and EB, and under CF and FD, are also equal (1); take away these from the equal squares, and the remainders, that is, the squares of GE and HF (3) are equal, and therefore the lines themselves GE and HF are equal (4); therefore the sums and differences of them and the halves, that is, the segments AE and CF, EB and FD, are equal.

Fig. 10.
(1) Hypoth.
(2) Cor. 1. Prop. 46. B. 1.
(3) Prop. 5.
(4) Cor. 2. Prop. 46. B. 1.

Cor. 3. The rectangle under the sum and difference of two right lines is equal to the difference between their squares.

Fig. 9.

Because that rectangle, together with the square of the less, is equal to the square of the greater, as is evident from the preceding proposition; for AC is the greater line, CD the less, and DB the difference.

Schol. 1. Hence it is evident that, in a right-angled triangle, the rectangle under the sum and difference of the hypothenuse and one side is equal to the square of the other side.

Schol. 2. Hence it is also evident that, if from any angle of a triangle a perpendicular be drawn to the opposite side, the rectangle under the sum and difference of the sides is equal to the rectangle under the sum and difference of the segments of the side, to which the perpendicular is drawn.

For the rectangle under the sum and difference of the sides is equal to the difference between the squares of the sides, and the rectangle under the sum and difference of the segments of the base is equal to the difference between the squares of the segments; and these differences are equal by Cor. 4. prop. 47. B. 1.

Fig. 11. Cor. 4. The difference between the squares of two sides BA and AC of any triangle BAC, is equal to double the rectangle under the remaining side BC, and the distance from its middle point D of the perpendicular AF, which is drawn to it from the opposite angle.

For the difference between the squares of the sides AB and AC is equal to the difference between the squares of the segments, BF and FC, of the side upon which the perpendicular falls (1); and therefore, when the perpendicular falls within the triangle, to the rectangle under BC the sum of these segments and the difference between BF and FC (2), or to double the rectangle under BC and half the difference; that is, the distance of the point F from the middle point of the side BC.

(1) Cor. 4. Prop. 47. B. 1.

(2) Cor. 3. Prop. 5.

But if the perpendicular fall without the triangle, the difference of the squares of the segments BF and FC is equal to the rectangle under BC the difference, and the sum of BF and FC; or to double the rectangle under BC and half the sum of BF and FC; that is, the distance of the point F from the middle point of the side BC.

Schol. Hence, given in numbers the sides of any triangle, we can find its area: divide the difference between the squares of two sides of the triangle by the remaining side; to half this side add half the quote, and subtract the square of this sum from the square of the greater side; the remainder is the square of the perpendicular, thence the perpendicular itself is found: which, multiplied into half the side upon which it falls, gives the area of the triangle.

PROP. VI. THEOR.

If a right line (AB) be bisected in (C), and produced to any point (F), the rectangle under the whole line thus produced (AF) and the produced part (BF), together with the square of the half (CB), is equal to the square of (CF) the line made up of the half and produced part. Fig. 13. See N.

On CF describe the square CEGF; draw EF, and through the point B draw BP parallel to FG, and cutting EF in K; through K draw LO parallel to CF, and meeting AO which is drawn through A parallel to CD.

Because AC and CB are equal (1), the rectangle AD (1) Hypoth.
is equal to the rectangle CK (2); but the rectangles CK (2) Prop. 36. B. 1.
and KG are equal (3), therefore AD is equal to KG, (3) Prop. 43.
add to both CL, and AL is equal to the gnomon CLP; B. 1.
add to both DP, and the sum of AL and DP is equal (4) Cor.
to the square of CF. But AL is the rectangle under Prop. 43. B. 1. & Def. 31.
the whole produced line and the produced part, for FL B. 1.
is equal to BF (4); and DP is the square of the half (5) Cor.
CB, for it is the square of DK (5), and DK is equal to Prop. 43. B. 1 (6) Prop. 31.
CB (6). B. 1.

Otherwise thus:

The rectangle under AF and BF is equal to the sum of the rectangles under AC and BF, and under CB and BF, together with the square of BF (1): but the rectangle under AC and BF is equal to the rectangle under CB and BF, as AC and CB are equal: therefore the rectangle under AF and BF is equal to double the rectangle under CB and BF, together with the square of BF: add to both the square of CB, and the rectangle AFB, together with the square of CB, is equal to double the rectangle CBF, together with the squares of CB and BF, that is, to the square of CF (2). Fig. 14. (1) Prop. 1. (2) Prop. 4.

Cor. If a right line BE be drawn from the vertex of an isosceles triangle to the base, or the base produced, Fig. 12.

the rectangle under the segments AE and EC of the base is equal to the difference between the square of this line BE, and the square of either side AB or BC.

Bisect AC in D, and draw the right line BD.

The rectangle AEC is equal to the difference between the squares of DC and of DE (1): add to both the square of the perpendicular BD, and the rectangle AEC is equal to the difference between the sum of the squares of DC and DB or the square of BC (2), and the sum of the squares of DE and DB or the square of EB (2).

(1) Prop. 5. & 6.
(2) Prop. 47. B. 1.

But if the line drawn to the base be perpendicular to it, it bisects the base, and the rectangle under the segments of the base is the square of the half; and therefore, together with the square of the line drawn to the base, equal to the square of the side (2).

PROP. VII. THEOR.

Fig. 15.

If a right line (AB) be divided into any two parts, the sum of the squares of the whole line (AB) and either segment (CB) is equal to double the rectangle under the whole line and that segment, together with the square of the other segment.

Describe the square of AB; draw FB; through the point C draw CG parallel to AF, and through P, its intersection with FB, draw DE parallel to AB.

The square AK is equal to the rectangles AE and PK, together with the square DG: add to both the square CE, and the squares AK and CE, taken together, are equal to the rectangles AE and CK together with the square DG.

But AE is equal to the rectangle under AB and CB, because CB and BE are equal (1); and CK is also equal to the rectangle under AB and CB, because KB is equal to AB (2); and DG is the square of AC, because DP and AC are equal (3).

(1) Cor. Prop. 43. B. 1.
(2) Def. 31. B. 1.
(3) Prop. 34. B. 1.

Otherwise thus :

The square of AB is equal to double the rectangle ACB, together with the squares of AC and CB (1); add to both the square of CB, and the square of AB, together with the square of CB, is equal to double the rectangle ACB together with the square of AC and double the square of CB, but double the rectangle ACB, with double the square of CB, is equal to double the rectangle ABC (2); therefore the sum of the squares of AB and CB is equal to double the rectangle ABC together with the square of AC.

Fig. 16. (1) Prop. 4. (2) Prop. 3.

Cor. Hence it is evident that the excess of the sum of the squares of two lines AB and CB, above the double rectangle under them ABC, is equal to the square of the difference between them AC. Fig. 16.

PROP. VIII. THEOR.

If a right line (AC) be divided into any two parts (in B), the square of the sum of the whole line (AC) and either segment (BC) is equal to four times the rectangle under the whole line (AC) and that segment (BC), together with the square of the other segment (AB). Fig. 17.

Produce AC till CD is equal to BC (1); on AD describe the square ARZD (2), and through the points B and C draw BS and CV parallel to AR; having drawn RD, draw, through the points G and K, EH and LP parallel to AD.

(1) Post. 1. & Prop. 3. B. 1. (2) Prop. 46. B. 1.

Because SV is equal to BC (3), BC to CD (4), and CD to VZ (3), SV and VZ are equal, and therefore the rectangles SG and VH are equal (5): but VH and AG are also equal (6), therefore SG is equal to AG; and because FG is equal to BC (3), FG and CD are equal, and therefore the square FO is equal to the square CH; and also EK and KV are equal (6), and if to these equals be added the equals CH and FO, EK and CH together shall be equal to SG, and therefore to AG; therefore

(3) Prop. 34. B. 1. (4) Constr. (5) Prop. 36. B. 1. (6) Prop. 43. B. 1.

AG, SG and VH, together with EK and CH are four times AG; but AG, SG and VH, together with EK, CH, and the square LS, are equal to the square AZ; therefore AG four times taken, together with LS, is equal to AZ.

(7) Cor. Prop. 43. B. 1. (8) Constr. (3) Prop. 34. B. 1.

But AG is the rectangle under AC and BC, because CG is equal to CD (7), and therefore to BC (8); and LS is the square of AB, because AB and RS are equal (3).

Otherwise thus:

Fig. 18.

Produce AC till CD is equal to BC: the square of AD is equal to the squares of AC and CD, together with double the rectangle ACD; that is, because BC and CD are equal (1), to the sum of the squares of AC and BC, together with double the rectangle ACB: but the sum of the squares of AC and BC is equal to double the rectangle ACB, together with the square of AB (2); therefore the square of AD is equal to the rectangle ACB four times taken, together with the square of AB.

(1) Constr. (2) Prop. 7.

PROP. IX. THEOR.

Fig. 19.

If a right line be divided into equal parts (in C) and into unequal (in D), the sum of the squares of the unequal parts (AD and DB) is equal to double the sum of the squares of the half (AC) and of the intermediate part (CD).

(1) Prop. 11. & 3. B. 1.

From the point C draw CE perpendicular to AB and equal to either AC or CB (1); join AE and EB; and through D draw DF parallel to CE, and through F draw FG parallel to CD; and join FA.

(2) Constr. (3) Cor. 3. Prop. 32. B. 1. (4) Prop. 29. B. 1.

Because the angle ACE is a right angle, and the sides AC and CE are equal (2), CEA is half a right angle (3); in the same manner it can be demonstrated that CEB is half a right angle; therefore AEB is a right angle: on account of the parallels GF and CD, the angle EGF is equal to ECB (4), therefore EGF is a right angle, but GEF is half a right angle, therefore GFE is also half a right angle, and therefore GE and GF are

equal (5); likewise FDB is a right angle, because it is equal to the angle ECB, on account of the parallels FD and CE: but DBF is half a right angle; therefore DFB is half a right angle, and therefore DF and DB are equal (5). Since therefore AC and CE are equal, and the angle ACE right, the square of AE is double the square of AC, and because EG and GF are equal, and the angle EGF right, the square of EF is double the square of GF, but GF and CD are equal (6); therefore the square of EF is double the square of CD, and therefore the squares of AE and EF are double the squares of AC and CD; but because the angle AEF is right, the square of AF is equal to the squares of AE and EF (7); therefore the square of AF is double the squares AC and CD; but the square of AF is equal to the squares of AD and DF, as the angle ADF is right (7), therefore the sum of the squares of AD and DF is double the sum of the squares of AC and CD; but DF and DB are equal, and therefore the sum of the squares of AD and DB is double the sum of the squares of AC and CD.

(5) Prop. 6. B. 1.

(6) Prop. 34. B. 1.

(7) Prop. 47. B. 1.

Otherwise thus:

The square of AD is equal to the squares of AC and CD, together with double the rectangle ACD (1), or, because AC and CB are equal, together with double the rectangle BCD; add to both the square of DB, and the squares of AD and DB are equal to the squares of AC, CD, and DB, together with double the rectangle BCD: but double the rectangle BCD, with the square of DB, is equal to the squares of CB and CD (2), or, because AC and CB are equal, to the squares of AC and CD, therefore the sum of the squares of AD and DB is equal to double the sum of the squares of AC and CD.

Fig. 20.

(1) Prop. 4.

(2) Prop. 7.

PROP. X. THEOR.

Fig. 21. *If a right line (AB) be bisected (in C) and produced to any point (D); the square of the whole line thus produced (AD), together with the square of the produced part (BD), is equal to double the square of the line (CD) made up of the half and produced part, together with double the square of half the given line (AC).*

From the point C draw CE perpendicular to AB,
(1) Prop. 11. & 3. B. 1. and equal to either CA or CB (1); join AE; and draw,
through the point E, EF parallel to AB (2), and
(2) Prop. 30. B. 1. through D, DF parallel to CE; because the angles
CEF and DFE are equal to two right angles on account
(3) Prop. 29. B. 1. of the parallel lines CE and DF (3), the angles BEF
and DFE are less than two right angles; therefore the
(4) Ax. 12. lines EB and FD, if produced, must meet (4): let them
meet in G and draw GA.

(5) Constr. Because CA and CE are equal (5), and the angle C
a right angle (5), the angle CEA, is half a right angle
(6) Cor. 3. Prop. 32. B. 1. (6): in the same manner it is proved that CEB is half
a right angle, therefore AEB is a right angle: and because DG and EC are parallel (5), the alternate angles
GDC, ECB are equal, therefore GDB is a right angle;
(7) Prop. 15. B. 1. also the angles DBG and EBC are equal (7), but EBC
is half a right angle, therefore DBG is half a right angle,
and also DGB, and therefore the sides DB and DG
(8) Prop. 6. B. 1. are equal (8); and because EGF is half a right angle,
and the angle at F right, being equal to its opposite C
(9) Prop. 34. B. 1. (9), FEG is half a right angle, and therefore the sides
EF and FG are equal.

Because AC and CE are equal, and the angle ACE right, the square of AE is double the square of AC; and because GF and FE are equal, and the angle F right, the square of GE is double the square of EF; but EF and CD are equal (9), therefore the square of GE is double the square of CD; the square of AE is also double the square of AC, therefore the squares of AE and EG are together double the squares of AC and CD; but the square of AG is equal to the squares of AE

and EG (10), and is therefore double the squares of AC and CD, and the squares of AD and DG are equal to the square of AG (10), and therefore double the squares of AC and CD; but BD and DG are equal, and therefore the squares of AD and DB are double the squares of AC and CD. (10) Prop. 47. B. 2.

Otherwise thus:

The square of AD is equal to the squares of AC and CD, together with double the rectangle ACD (1), or together with double the rectangle DCB, because AC and CB are equal. But double the rectangle DCB, together with the square of DB, is equal to the squares of DC, and CB or AC (2). Therefore the square of AD, together with the square of DB, is equal to double the square of AC and double the square of CD. Fig. 22. (1) Prop. 4. (2) Prop. 7.

Cor. The sum of the squares of two lines is equal to double the square of half their sum and double the square of half their difference. For CD is half the sum of the lines AD and BD, and AC is half their difference. Fig. 20, 21.

PROP. XI. PROB.

To divide a given finite right line (AB), so that the rectangle under the whole line and one segment shall be equal to the square of the other segment. Fig. 23.

From the point A erect AC perpendicular and equal to the given line AB (1); bisect it in E (2): join EB; produce CA until EF is equal to EB, and in the given line AB take AH equal to AF; the square of AH is equal to the rectangle under the other segment HB and the whole line AB. (1) Prop. 11. & 3. B. 1. (2) Prop. 10. B. 1.

Complete the square of AB; draw through H the right line GK parallel to AC, and through F the right line FG parallel to AB.

Because CA is bisected in E, and AF is added to it, the rectangle under CF and FA, together with the

(1) Prop. 6. (2) Constr. (3) Prop. 47. B. 1. (4) Constr. & Prop. 34. B. 1.

square of EA is equal to the square of EF (1), or to the square of EB which is equal to EF (2), and therefore to the squares of EA and AB (3), take away the common square of EA, and the rectangle under CF and FA is equal to the square of AB; but, because AF and FG are equal (4), CG is the rectangle under CF and FA, therefore CG and AD are equal; and, if the common rectangle CH be taken away, AG and HD are equal; but AG is the square of AH, for AH and AF are equal (2), and the angle A is a right angle; HD is the rectangle under AB and HB, for BD is equal to AB.

Cor. The rectangle under the greater segment and the difference between the segments is equal to the square of the less segment; as is evident by taking away the rectangle under AH and HB from the equals AG and HD.

PROP. XII. THEOR.

Fig. 24.

In any obtuse-angled triangle (BAC) the square of the side (AB), subtending the obtuse angle, exceeds the sum of the squares of the sides (BC and CA) which contain the obtuse angle, by double the rectangle under either of these sides (BC), and the external segment (CD) between the obtuse angle and the perpendicular drawn from the opposite angle.

(1) Prop. 47. B. 1. (2) Prop. 4.

The square of AB is equal to the sum of the squares of AD and DB (1); but the square of DB is equal to the squares of DC and CB, together with double the rectangle under DC and CB (2); therefore the square of AB is equal to the squares of AD, DC and CB, together with double the rectangle under DC and CB; but the square of AC is equal to the squares of AD and DC (1), and therefore the square of AB is equal to the squares of AC and CB, together with double the rectangle under BC and CD; therefore the square of AB exceeds the sum of the squares of AC and CB by double the rectangle under DC and CB.

PROP. XIII. THEOR.

In any triangle (ABC) the square of the side (AB), subtending an acute angle, is less than the sum of the squares of the sides (AC and CB) containing that angle, by twice the rectangle under either of them (AC) and the segment between the acute angle and the perpendicular (BF), let fall from the opposite angle. Fig. 25.

The squares of AC and CF are equal to twice the rectangle under AC and CF, together with the square of AF (1); and, if the square of the perpendicular BF be added to both, the squares of AC, CF and BF are equal to twice the rectangle under AC and CF, together with the squares of BF and AF, or with the square of AB, which is equal to them (2); but the squares of BF and CF are equal to the square of BC (2), and therefore the squares of BC and AC are equal to twice the rectangle under AC and CF, together with the square of AB: therefore the square of AB is less than the sum of the squares of AC and CB, by twice the rectangle under AC and CF.

(1) Prop. 7.
(2) Prop. 47. B. 1.

Schol. 1. If the angle CAB be a right angle, the points A and F coincide, and the rectangle under AC and CF is the square of AC; but it is evident that in this case the square of AB is less than the sum of the squares of AC and CB by twice the square of AC (1).

(1) Prop. 47. B. 1.

Schol. 2. Hence, given in numbers the sides of any triangle, we can find its area; for subtract the square of one of the sides AB, which is not the greatest, from the sum of the squares of the other sides AC and CB, and divide half the remainder (that is, the rectangle under AC and CF) by either of these sides, AC, and subduct the square of the quote CF from the square of the remaining side CB; the square root, BF, of the remainder multiplied into half the former divisor AC gives the area of the triangle.

Cor. If from any angle A, of a triangle BAP, a right line AC be drawn bisecting the opposite side, the squares of the sides AB and AP, containing that angle, are Fig. 26.

double of the squares of the bisecting line AC, and of half the side subtending the angle.

If the bisecting line be perpendicular to the side, it is manifest from Prop. 47. B. 1.

If not, draw from the angle A the line AF perpendicular to the opposite side; since one of the angles ACB and ACP is obtuse, let ACB be the obtuse angle, and the square of AB is equal to the sum of the squares of AC and CB, together with twice the
(1) Prop. 12. rectangle under BC and CF (1): the angle ACP is acute, and therefore the square of AP, together with twice the rectangle under PC and CF, or (as PC
(2) Hypoth. and BC are equal (2) with twice the rectangle under BC and CF, is equal to the sum of the squares of
(3) Prop. 13. AC and CP (3); therefore the squares of AB and AP, with twice the rectangle under BC and CF, are equal to twice the square of AC, with the squares of BC and CP, and twice the rectangle under BC and CF: take away from both twice the rectangle under BC and CF, and the sum of the squares of AB and AP is equal to twice the square of AC, with the squares of BC and CP, or with twice the square of BC; because CP and BC are equal.

Fig. 7. Schol. Hence it is evident that the sum of the squares of the sides of a parallelogram is equal to the
(1) Prop. 29. B. 1. sum of the squares of the diagonals.

(2) Prop. 34. B. 1. Because the angles DBC and BDA, ACB and CAD are equal (1), and the side BC is equal to AD (2), the
(3) Prop. 4. B. 1. diagonals mutually bisect each other (3), therefore the
(4) Cor. pr. sum of the squares of the sides is equal to four times
(5) Cor. Prop. 4. the squares of the halves of the diagonals (4), or to the sum of the squares of the diagonals (5).

PROP. XIV. PROB.

Fig. 28. *To make a square equal to a given rectilineal figure (Z).*

Make a rectangle CI equal to the given rectilineal
(1) Prop. 45. B. 1. figure (1): if the adjacent sides be equal, the problem is done.

(2) Prop. 3. B. 1. If not, produce either side IA and make the produced part AL equal to the adjacent side AC (2);

bisect IL in O, and from the centre O, with the radius OL, describe a semicircle LBI, and produce CA till it meet the periphery in B: the square of AB is equal to the given rectilineal figure.

For draw OB, and because IL is bisected in O and cut unequally in A, the rectangle under IA and AL, together with the square of OA, is equal to the square of OL (3), or of OB which is equal to OL, and therefore to the squares of OA and AB (4); take away from both the square of OA, and the rectangle under IA and AL is equal to the square of AB: but the rectangle under IA and AL is equal to IC, for AL and AC are equal (5); therefore the square of AB is equal to the rectangle IC, and therefore to the given rectilineal figure Z.

(3) Prop. 5.
(4) Prop. 47. B. 1.
(5) Constr.

THE

ELEMENTS OF EUCLID.

BOOK III.

DEFINITIONS.

See N. 1. EQUAL circles are those, whose diameters are equal.

Plate 3. Fig. 1. 2. A right line is said to touch a circle, when it meets the circle, and being produced does not cut it.

3. Circles are said to touch one another, which meet, but do not cut one another.

4. Right lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.

5. And the right line, on which the greater perpendicular falls, is said to be farther from the centre.

Fig. 2. 6. A segment of a circle is the figure contained by a right line, and the part of the circumference it cuts off.

Fig. 3. 7. An angle in a segment is the angle contained by two right lines drawn, from any point in the circumference of the segment, to the extremities of the right line, which is the base of the segment.

8. An angle is said to stand upon the part of the circumference, or the arch, intercepted between the right lines that contain the angle.

Fig. 4. 9. A sector of a circle is the figure contained by two radii and the arch between them.

See N. 10. Similar segments of circles are those, which contain equal angles.

PROP. I. PROB.

To find the centre of a given circle (ACB). Fig. 5.

Draw within the circle any right line AB, bisect it in D (1); from D draw DC perpendicular to AB (2), and produce it to E; bisect CE in F, and F is the centre. (1) Prop. 10. B. 1. (2) Prop. 11. B. 1.

For, *if it be possible*, let any other point G be the centre; and draw GA, GD, and GB.

Because in the triangles GDA, GDB, the side GA is equal to GB (3), DA equal to DB (4), and the side GD common to both, the angles GDA and GDB are equal (5), and therefore are right angles (6); but the angle CDB is a right angle (4), therefore GDB is equal to CDB (7), a part equal to the whole, which is absurd: G therefore is not the centre of the circle ACB; and in the same manner it can be proved that no other point without the line CE is the centre; therefore the centre is in the line CE, and therefore is the point F. (3) Hypoth. & Def. 15. B. 1. (4) Constr. (5) Prop. 8. B. 1. (6) Def. 11. B. 1. (7) Ax. 11.

Schol. Hence it is evident that, if any line terminated in a circle be bisected by a perpendicular, that perpendicular, if produced, will pass through the centre.

PROP. II. THEOR.

If any two points (A and B) be taken in the circumference of a circle, the right line which joins them falls within the circle. Fig. 6. See N.

For, *if it be possible*, let AEB be a right line, in which the point E is without the circle; and draw DA, DE, and DB.

Because in the triangle ADB, the sides DA and DB are equal (1), the angle DBA is equal to DAB (2); but the external angle DEA is greater than the internal angle DBA (3), therefore greater than the angle DAB, and therefore the side DA is greater than the side DE (4); but the right line DF is equal to DA (1), and (1) Def. 15. B. 1. (2) Prop. 5. B. 1. (3) Prop. 16. B. 1. (4) Prop. 19. B. 1.

therefore is greater than DE, a part greater than the whole, which is absurd; therefore the line AEB is not a right line; and in the same manner it can be demonstrated that if the point E be in the circumference, the line is not a right line.

Therefore every point of a right line falls within the circle.

PROP. III. THEOR.

Fig. 7. *If a right line (BL), drawn through the centre of a circle, bisect a right line (CF), which does not pass through the centre, it is perpendicular to it.*

And if it cut it at right angles, it bisects it.

(1) Def. 15. B. 1. (2) Hypoth. (3) Prop. 8. B. 1. (4) Def. 11. B. 1.

Part 1. Draw AC and AF; in the triangles AOC, AOF, the side AC is equal to AF (1), and also OC to OF (2), and AO is common to both; therefore the angle AOC is equal to AOF (3), therefore each of them is a right angle (4), and therefore BO is perpendicular to CF (4).

(5) Def. 15. B. 1. (6) Prop. 26. B. 1.

Part 2. Because the triangle FAC is isosceles (1), the angle AFC is equal to the angle ACF (5); therefore in the triangles CAO, FAO, the angles ACO and AFO are equal, also AOC and AOF are equal (2), and the side AO opposite to the equal angles ACO, AFO, is common to both; therefore the side OC is equal to OF (6), and therefore the right line CF is bisected.

PROP. IV. THEOR.

Fig. 8. *If in a circle two right lines cut one another, and do not both pass through the centre, they do not bisect one another.*

If one of the lines pass through the centre, it is evident that it cannot be bisected by the other, which does not pass through the centre.

But if neither of the lines BC and FL pass through the centre, draw OA from the centre to their intersec-

tion; if BC be bisected in A, OA is perpendicular to it (1), and therefore the angle OAC is a right angle; and, if FL be bisected in A, OA is perpendicular to FL (1); therefore the angle OAL is a right angle, and therefore equal to the angle OAC (2), a part equal to the whole, which is absurd; therefore the lines BC and FL do not bisect one another.

(1) Prop. 2.
(2) Ax. 11.

PROP. V. THEOR.

If two circles (ABC, ABF) cut one another, they have not the same centre. Fig. 9.

For, *if it be possible*, let A be the centre of both circles; and draw two right lines, the one AF cutting both circles in C and F, the other AB to the intersection B.

Because A is the centre of the circle ABC, AB is equal to AC (1), and because A is the centre of the circle ABF, AB is equal to AF (1); therefore AC is equal to AF (2), a part to the whole, which is absurd; A therefore is not the centre of both circles; and in the same manner it can be proved, that no other point is the centre of both.

(1) Def. 15. B. 1.
(2) Ax. 1.

PROP. VI. THEOR.

If two circles (ABC, ABF) touch one another internally, they have not the same centre. Fig. 10.

For, *if it be possible*, let A be the centre of both circles, and draw two right lines, the one AF cutting both circles in C and F, the other AB to the point of contact.

Because A is the centre of the circle ABC, AB is equal to AC (1); and because A is the centre of the circle ABF, AB is equal to AF (1); therefore AC is equal to AF (2), a part equal to the whole, which is absurd: therefore the point A is not the centre of both

(1) Def. 15. B. 1.
(2) Ax. 1.

circles, and in the same manner it can be demonstrated that no other point is.

PROP. VII. THEOR.

Fig. 11. See N. *If, from any point within a circle, which is not the centre, right lines be drawn to the circumference, the greatest is that which passes through the centre.*

The remaining part of the diameter is the least.

Those lines, which make equal angles with the diameter, are equal.

That line, which is nearer to the line passing through the centre, is greater than one more remote.

And more than two right lines cannot be drawn, which shall be equal.

Part 1. The line CB passing through the centre is greater than any other CD.

Draw from the centre A the line AD; AB is equal to AD (1), therefore if CA be added to both, CB shall be equal to CA and AD taken together; but CA and AD are greater than CD (2), and therefore CB is greater than CD.

(1) Def. 15. B. 1. (2) Prop. 20. B. 1.

Part 2. The other part of the diameter CF is less than any other line CE.

Draw AE; AC and CE taken together are greater than AE (2), and therefore greater than AF, take away the common line AC from both, and CE is greater than CF.

Part 3. The right lines CL and CD, which make equal angles with the line CB passing through the centre are equal.

For, *if possible*, let one of them CL be the greater, and make CG equal to CD, and draw AD and AG.

In the triangles ACG and ACD, the side AC is common to both, CG is equal to CD (4), and the angles ACG and ACD are equal (5), therefore the sides AG and AD are equal (6); but AD is equal to AO, and therefore AG is equal to AO, a part equal to the whole, which is absurd. Therefore neither CL nor CD is greater than the other, and therefore they are equal.

(4) Constr. (5) Hypoth. (6) Prop. 4. B. 1.

Part 4. The line CD, or CL, which is nearer to the line passing through the centre, is greater than one more remote, CE.

If the given lines CD and CE be at the same side of CB, draw AD and AE; in the triangles CAD, CAE, the sides CA and AD are equal to CA and AE, and the angle CAD is greater than CAE, therefore the side CD is greater than CE (7). (7) Prop. 24. B. 1.

But if the given lines CL and CE be at different sides of CB, construct the angle ACD equal to ACL, and CD shall be equal to CL (8); but CD is greater than CE, and therefore CL is greater than CE. (8) Part 3.

Part 5. More than two right lines cannot be drawn, which shall be equal.

For let any three right lines be drawn from the point C to the circumference, and either one of them shall be part of a diameter, and therefore greater or less than either of the others (by part first and second); or two of them must be at the same side of the diameter, and therefore unequal (by part fourth).

PROP. VIII. THEOR.

If, from any point without a circle, lines be drawn to the circumference, those which make equal angles with the line passing through the centre are equal. Fig. 12, 13.

Of those lines, which are incident upon the concave circumference, the greatest is that which passes through the centre.

Of the rest that, which is nearer to the line passing through the centre, is greater than the more remote.

But of those incident upon the convex circumference, that line is the least which, if produced, would pass through the centre.

Of the rest that, which is nearer to the least, is less than the more remote.

Only two lines can be drawn, either to the concave or convex circumference, which shall be equal.

Part 1. The right lines AB and AX, which make equal angles with AZ, are equal.

For, *if it be possible*, let one of them AB be greater than the other; make AE equal to AX, and draw ZE and ZX.

In the triangles ZAE, ZAX, the side ZA is common, AE is equal to AX (1), and the angle ZAE is equal to ZAX (2), therefore the sides ZE and ZX are equal (3); but the line ZO is equal to ZX (4); therefore ZE is equal to ZO, a part equal to the whole, which is absurd. Therefore neither AB nor AX is greater than the other, and therefore they are equal.

(1) Constr.
(2) Hypoth.
(3) Prop. 4. B. 1.
(4) Def. 15. B. 1.

Fig. 12.

Part 2. Of those lines, which are incident upon the concave circumference, that line AY which passes through the centre is greater than any other AX.

Draw ZX, and ZY is equal to ZX (4); therefore, if AZ be added to both, AY shall be equal to AZ and ZX taken together; but AZ and ZX together are greater than AX (5), therefore AY is greater than AX.

(5) Prop. 20. B. 1.

Part 3. The line AB, or AX, which is nearer to the greatest, is greater than the more remote AD.

If the given lines AX and AD be at the same side of AY, draw ZX and ZD; in the triangles AZX, AZD the sides AZ, ZX, are equal to the sides AZ, ZD, and the angle AZX is greater than AZD, therefore the side AX is greater than AD (6).

(6) Prop. 24. B. 1.

But if the given lines AB and AD be at different sides of AY, make the angle ZAX equal to ZAB; and AX shall be equal to AB (7); but AX is greater than AD, therefore AB is greater than AD.

(7) Part. 1.

Fig. 13.

Part 4. Of those lines incident on the convex circumference, that line AF which, if produced, would pass through the centre is less than any other AX.

Draw ZF and ZX; ZX and XA together are greater than ZA (5), and therefore, if the equals ZX and ZF be taken away, AX is greater than AF.

Part 5. That line AB, or AX, which is nearer to the least, is less than the more remote AC.

If the given lines AX and AC be at the same side of AZ, draw ZX and ZC; ZC and CA taken together are greater than ZX and XA (8); take away the equals ZC and ZX, and AC is greater than AX.

(8) Prop. 21. B. 1.

But if the given lines AB and AC be at different sides of AZ, make the angle ZAX equal to ZAB, and

AX shall be equal to AB (7); but AC is greater than AX, and therefore greater than AB. (7) Part. 1.

Part. 6. Only two equal lines can be drawn either to the concave or convex circumference.

If any three lines be drawn, either one of them shall pass through the centre, and therefore be greater or less than either of the others (9); or two must be at the same side of the line passing through the centre, and therefore unequal (10). (9) Part. 2, & 4. (10) Part 3. & 5.

Schol. As it is evident that any right line drawn to the convex circumference is less than any line drawn to the concave, it follows that, if any three lines be drawn from a point without a circle to its circumference, only two of them can be equal.

Cor. Hence, and from part 5. prop. 7. it is evident that there is no point except the centre, from which three equal right lines can be drawn to the circumference of a circle.

PROP. IX. THEOR.

If a point be taken within a circle, from which more than two equal right lines can be drawn to the circumference, that point is the centre of the circle.

For, if it were a point different from the centre, only two equal right lines could be drawn from it to the circumference (1). (1) Prop. 7.

PROP. X. THEOR.

One circle (BDF) cannot cut another (BLF) in more than two points. Fig. 14, 15, See N.

For, *if it be possible*, let it cut the other in three points B, F, and C; let A be the centre of the circle BLF, and draw from it to the points of intersection the lines AB, AF, and AC; these lines are equal (1): but as the circles intersect, they have not the same (1) Def. 15. B. 1.

(2) Prop. 5. centre (2); therefore A is not the centre of the circle BDF, and therefore, as three right lines AB, AF, AC, are drawn from a point not the centre, these lines are (3) Cor. Prop. 8. not equal (3); but it was shewn before that they were equal, which is absurd; the circles therefore do not intersect in three points.

Schol. Hence it is evident that one circle cannot meet another in more than two points.

PROP. XI. THEOR.

Fig. 16. See N. *If two circles (ECF and DCL) touch one another internally, the right line joining their centres, being produced, shall pass through a point of contact.*

For, *if it be possible*, let A be the centre of the circle ECF, B the centre of the circle DCL; let DL be the line joining the centres, and from C a point of contact draw the lines CB and CA.

Because in the triangle BAC, the sides BA and AC together (1) Prop. 20. B. 1. are greater than BC (1), and BC is equal to BD, as they are radii of the circle DCL, the lines BA and AC are greater than BD; take away BA, which is common to both, and AC shall be greater than AD; but AC is equal to AE, because they are radii of the circle ECF, and therefore AE is greater than AD, a part greater than the whole, which is absurd. The centres are not, therefore, so placed that a line joining them can pass through any point of contact.

PROP. XII. THEOR.

Fig. 17. *If two circles (AOC and BFC) touch one another externally, the right line joining their centres passes through a point of contact.*

For, *if it be possible*, let A and B be the centres, and let the right line AB joining them not pass through a

point of contact; and from C a point of contact draw CA and CB to the centres.

Because in the triangle ACB, the sides AC and CB together are greater than AB (1), and the line AO is equal to AC, as they are radii of the circle AOC, and the line BF is equal to BC as they are radii of the circle BFC, AO and BF taken together are greater than BA, a part greater than the whole, which is absurd. The centres are not therefore so placed that the line joining them can pass through any point but a point of contact. (1) Prop. 20. B. 1.

PROP. XIII. THEOR.

One circle cannot touch another, either externally or internally, in more points than one. Fig. 18, 19, 20. See N.

For, *if it be possible*, let the circles ADE and BDF touch one another internally in two points D and C; draw the line AB joining their centres, and produce it until it pass through one of the points of contact D, and draw AC, BC.

Because BD and BC are radii of the circle BDF, BD is equal to BC, therefore if AB be added to both, AD shall be equal to AB and BC together; but AD and AC are radii of the circle ADE, therefore AD is equal to AC, and therefore AB and BC are equal to AC; but they are greater than it (1), which is absurd. (1) Prop. 20. B. 1.

But if the points of contact be at the extremities of the right line joining the centres, CD must be bisected in A and also in B, because it is a diameter of both circles; which is absurd. Fig. 19.

Next, *if it be possible*, let the two circles ADE and BDF touch one another externally in two points D and C; draw the right line AB, joining the centres of the circles, and passing through one of the points of contact C, and draw AD and DB. Fig. 20.

Because AD and AC are radii of the circle ADE, they are equal; and because BC and BD are radii of the circle BDF, they also are equal; therefore AD and BD together are equal to AB; but they are greater than it (1), which is absurd.

There is therefore no case, in which two circles can touch one another in two points.

Schol. Hence it is evident that a circle touching another circle never can meet it again.

PROP. XIV. THEOR.

Fig. 21. *In a circle, equal right lines (BC and FL) are equally distant from the centre.*

And right lines (BC and FL), which are equally distant from the centre, are equal.

Let A be the centre of the circle; join AC, AL, and draw AO and AI perpendicular to BC and FL.

(1) Hypoth. Part 1. Because BC and FL are equal (1), and
(2) Prop. 3. perpendiculars from the centre bisect them (2), OC and
(3) Ax. 7. (4) Cor. 1. IL are equal (3), and therefore their squares are equal
Prop. 46. (4); but AC and AL are also equal, and therefore their
B. 1. squares are equal (4); but the square of AC is equal to
(5) Prop. 47. B. 1. the squares of AO and OC (5), and the square of AL
is equal to the squares of AI and IL (5); therefore the
squares of AO and OC are equal to the squares of AI
(6) Ax. 3. (7) Cor. 2. and IL; take away the equal squares of OC and IL, and
Prop. 46. the squares of AO and AI are equal (6), and therefore
B. 1. the lines themselves are equal (7).

Part 2. Because AO and AI are equal (1), their squares are equal (4); but AC and AL are equal, and therefore their squares are equal (4); but the square of AC is equal to the squares of AO and OC (5); and the square of AL is equal to the squares of AI and IL (5); therefore the squares of AO and OC are equal to the squares of AI and IL: take away the equal squares of AO and AI, and the squares of OC and IL are equal (6), therefore the lines themselves are equal (7); but, because AO and AI bisect BC and FL (2), OC and IL are the halves of BC and FL; and, since they are equal,
(8) Ax. 6. the lines BC and FL are also equal (8).

PROP. XV. THEOR.

The diameter is the greatest right line in a circle: and of all others that which is nearer to the centre is greater than the more remote. Fig. 22. See N.

Part. 1. The diameter AB is greater than any other line ED.

For draw CD and CE; CD is equal to CB, and CE
to CA (1), therefore AB is equal to CD and CE toge- (1) Def. 15.
ther; but CD and CE together are greater than ED B. 1.
(2), and therefore AB is greater than ED. (2) Prop. 20. B. 1.

Part. 2. That which is nearer the centre is greater than one more remote.

First, let the given lines be ED and IK, which are at the same side and do not intersect; draw CD, CE, CI, and CK.

In the triangles ECD, ICK, the sides EC, CD, are
equal to IC, CK; but the angle ECD is greater than (3) Prop. 24.
ICK, therefore the side ED is greater than IK (3). B. 1.

Let the given lines be ZX and IK, which either are at different sides, or intersect; draw CO and CF perpendicular to ZX and IK, and from the greater CF
cut off CV equal to the less CO (4), and through V (4) Prop. 3.
draw ED perpendicular to CF. B. 1.

Because ZX and ED are equally distant from the
centre (5), ED is equal to ZX (6); but ED is greater (5) Constr.
than IK, and therefore ZX is greater than IK. (6) Prop. 14.

PROP. XVI. THEOR.

[redacted] *right line, drawn from the extremity of the di-* Fig. 23.
[redacted] *of a circle perpendicular to it, falls without the* See N.
[redacted] *if any right line be drawn from a point within*
[redacted] *pendicular to the point of contact, it cuts the*
[redacted]

Part 1. For, *if it be possible*, let BG be perpendicular to AB, and meet the circle again in G; and draw CG.

Because in the triangle CGB the side GC is equal to (1) Prop. 5.
CB, the angle CBG, is equal to CGB (1), and therefore B. 1.

(2) Cor. Prop. 17. B. 1.
(3) Hypoth.

each of them is acute (2); but CBG is a right angle (3), which is absurd. Therefore the right line drawn through B perpendicular to AB does not meet the circle again.

Part 2. Let BF be perpendicular to AB, and let EB be a line drawn from a point between it and the circle, which, *if it be possible*, does not cut the circle.

(4) Cor. 1. Prop. 16. B. 1.

Because the angle CBF is a right angle, CBE is acute (2); draw CI perpendicular to BE, and it must fall at the side of the angle CBE (4).

(5) Prop. 19. B. 1.

Then in the triangle BCI the angle CIB is greater than CBI, therefore the side CB is greater than CI (5); but CO is equal to CB, and therefore CO is greater than CI, a part greater than the whole, which is absurd. Therefore the point I does not fall outside the circle, and therefore the right line BE cuts the circle.

Schol. 1. Hence it is evident that the right line BF touches the circle in the point B, and that the tangent can only meet the circle in one point, and that at each point of the circumference there is only one tangent.

See N.

Schol. 2. It is also evident that the right line, which makes at B an acute angle however great, must meet the circle again.

Fig. 23.

Cor. 1. From this proposition is immediately deduced a method of drawing a tangent through any given point B in the circumference of a circle; draw through that point a diameter AB, and erect at the extremity of it a perpendicular BF.

Fig. 25.

Cor. 2. If the line CD be produced beyond its extremity C, and there be taken in the produced part any number of points, and from these points as centres circles be described through the point D, it is evident that each of these circles must touch the line in the point D, and cannot meet each other in any other point; whence the right line EF is cut by each of these circles in a different point, and is therefore *infinitely divisible*.

PROP. XVII. PROB.

From a given point (A), without a given circle (CBF), to draw a right line which shall be a tangent to the circle. Fig. 24.

Let C be the centre of the given circle; and from the centre C, with the radius CA, describe a circle CAE: draw CA meeting the circle in the point F, and draw through the point F the line FE perpendicular to CA and meeting the circle CAE in E; draw the line CE meeting in B the given circle; and the right line drawn from B to the given point A is a tangent.

For in the triangles ACB, ECF, the sides AC and CB are equal to EC and CF (1), and the angle at C is common to both, therefore the angle ABC is equal to EFC (2); but the angle EFC is a right angle (3); therefore ABC is a right angle, and therefore the right line AB is a tangent to the circle CFB (4).

(1) Def. 15. B. 1.
(2) Prop. 4. B. 1.
(3) Constr.
(4) Prop. 16.

Schol. It is evident that there can be drawn two tangents from the point A, one at either side of the right line AC.

PROP. XVIII. THEOR.

If a right line (DB) be a tangent to a circle, the right line (CD), drawn from the centre to the point of contact, is perpendicular to it. Fig. 25.

For, *if it be possible*, let the right line CF be perpendicular to BD; and in the triangle CFD, because the angle CFD is a right angle, the angle CDF is acute (1), therefore the side CD is greater than the side CF (2); but CE is equal to CD (3), and therefore CE is greater than CF, a part greater than the whole, which is absurd. Therefore CF is not perpendicular to BD; and in the same manner it can be demonstrated that no other line, except CD, is perpendicular to it.

(1) Cor. Prop. 17. B. 1.
(2) Prop. 19. B. 1.
(3) Def. 15. B. 1.

PROP. XIX. THEOR.

Fig. 26. *If a right line (BC) be a tangent to a circle, the right line (BA), drawn perpendicular to it from the point of contact, passes through the centre of the circle.*

For, *if it be possible*, let the centre Z be without the line BA; and draw ZB.

Because the right line ZB is drawn from the centre to the point of contact, it is perpendicular to the tan-
(1) Prop. 18. gent (1), therefore the angle ZBC is a right angle; but
(2) Hypoth. the angle ABC is also a right angle (2); and therefore ZBC is equal to ABC, a part to the whole, which is absurd. Therefore Z is not the centre; and in the same manner it can be demonstrated that no other point, without the line AB, is the centre.

PROP. XX. THEOR.

Fig. 27, 28, 29. See N. *The angle (ACD) at the centre of a circle is double of the angle (ABD) at the circumference, when they have the same part of the circumference for their base.*

Fig. 27. 1. Let one leg of the angle at the circumference pass through the centre; because, in the triangle DCB, the
(1) Prop. 5. B. 1. sides DC and CB are equal, the angles CBD and CDB
(2) Prop. 32. B. 1. are equal (1); but the external angle ACD is equal to both of them taken together (2), and therefore is double of ABD.

Fig. 28. 2. Let the angle ACD fall within the angle ABD; and draw through the centre the right line BCE; be-
(3) Part 1. cause the angle ACE is double of the angle ABC (3), and the angle DCE is double of the angle DBC (3); the whole angle ACD is equal to double the angle ABC and double the angle DBC, and therefore equal to double the angle ABD.

Fig. 29. 3. Let one side of the angle ACD cut a leg of the angle ABD; and draw through the centre the right line BCE: the angle ECD is equal to double the angle

EBD (3), or to double the angle EBA together with double the angle ABD; but the angle ECA is equal to double the angle EBA (3); take away these equal quantities from both, and the angle ACD shall be equal to double the angle ABD (4).

(3) Part 1.
(4) Ax. 3.

PROP. XXI. THEOR.

The angles (BAD, BED) in the same segment of a circle are equal. Fig. 30, 31.

1. Let the segment BAD be greater than a semicircle; let C be the centre of the circle; and draw CB and CD. Fig. 30.

The angle BCD at the centre is double of the angle BAD (1), and also double of BED (1); therefore BAD and BED are equal to one another (2).

(1) Prop. 20.
(2) Ax. 7.

2. Let the segment BAD be a semicircle or less than a semicircle; let C be the centre of the circle; and draw the right lines ACF and EF. Fig. 31.

Because the segment BDF is greater than a semicircle, and in it are the angles BAF and BEF, BAF is equal to BEF (3); and because the segment FBAD is greater than a semicircle, and in it are the angles FAD and FED, FAD is equal to FED (3); therefore the sum of the angles BAF and FAD, or the angle BAD, is equal to the sum of BEF and FED (4), or to the angle BED.

(3) Part 1.
(4) Ax. 2.

Cor. If two equal angles stand upon the same arch BD, and the vertex of one of them BAD be in the circumference of the circle, the vertex of the other must be in the same circumference. Fig. 32.

For, *if it be possible*, let the vertex of the other angle fall either without or within the circumference, as at F, and draw ED; because the angles BAD and BED are in the same segment, they are equal (1); but BAD is equal to BFD (2), and therefore BED is equal to BFD; but one of them is greater than the other (3), which is absurd: therefore the point F neither falls within or without the circle, and therefore it falls upon the circumference itself.

(1) Prop. 21.
(2) Hypoth.
(3) Prop. 16. B. 1.

PROP. XXII. THEOR.

Fig. 33. *The opposite angles of a quadrilateral figure (FABC), inscribed in a circle, are together equal to two right angles.*

Draw the diagonals AC and FB.

Because the angles ACB and AFB are in the same segment AFCB, ACB is equal to AFB (1); and because the angles ACF and ABF are in the same segment ABCF, ACF is equal to ABF (1); therefore the angle BCF is equal to the angles AFB and ABF taken together; but the angles AFB and ABF together with FAB are equal to two right angles (2) and therefore BCF together with FAB is equal to two right angles: in the same manner it can be demonstrated that ABC and AFC are equal to two right angles.

(1) Prop. 21. (2) Prop. 32. B. 1.

Cor. If one of the sides of a quadrilateral figure inscribed in a circle be produced, the external angle is equal to the internal remote angle; for each of them together with the internal adjacent angle is equal to two right angles (1).

(1) Prop. 13. B. 1. & Prop. prec.

PROP. XXIII. THEOR.

Fig. 34. *Upon the same right line, and upon the same side of it, two similar segments of circles cannot be constructed which do not coincide.*

For, *if it be possible*, let two similar segments ACB and ADB be constructed, and let the point D in one of them fall without the other, and draw the right lines DA, DB and CB.

Because the segments ACB and ADB are similar, the angle ACB is equal to ADB (1); but ACB is external to ADB and therefore greater than it (2), which is absurd: therefore no point in either of the segments falls without the other, and therefore the segments coincide.

(1) Def. 10. & Prop. 21. (2) Prop. 16. B. 1.

PROP. XXIV. THEOR.

Similar segments of circles, standing upon equal right lines (AB and CD), are equal. Fig. 35.

For, if the equal right lines AB and CD be so applied to one another that the point A may fall on C, the point B must fall upon D; and therefore the right lines coincide (1): therefore the segments themselves coincide (2), and therefore they are equal. (3) Ax. 10. (2) Prop. 23.

PROP. XXV. PROB.

A segment (ABC) of a circle being given, to describe the circle of which it is the segment. Fig. 36. See N.

From any point B draw two right lines BA and BC; bisect them, and from the points of bisection F and E draw two lines FO and EO, perpendicular to AB and BC: the intersection O of these perpendiculars is the centre.

Because the right line AB terminated in the circle is bisected by a perpendicular to it FO, FO passes through the centre (1); likewise EO passes through the centre (1); therefore the centre must be in O the intersection of these lines FO and EO. (1) Schol. Prop. 1.

PROP. XXVI. THEOR.

In equal circles (ABC, DEF) equal angles (AOC and DHF, ABC and DEF), whether they be at the centres or at the circumferences, stand upon equal arches. Fig. 37. See N.

First, let the given angles AOC and DHF be at the centres; draw to any points B and E in the circumferences the lines AB, CB, and DE, FE; and join AC and DF.

Because in the triangles AOC, DHF, the angles O and H are equal (1), and the sides AO and OC equal to DH and HF (1), the bases AC and DF are equal (2): but the angles ABC and DEF are equal (3), and therefore the segments ABC and DEF are similar (4); but they stand upon equal right lines AC and DF, and are therefore equal (5); take away these equals from the (1) Hypoth. (2) Prop. 4. B. 1. (3) Prop. 20. et Ax. 7. (4) Def. 10. (5) Prop. 24.

equal circles, and the remaining segments are equal; and therefore the arches AGC and DKF are equal.

In the same manner it can be demonstrated that the arches AGC and DKF are equal, if the given angles at the circumferences ABC and DEF be acute; by drawing OA and OC, and also HD, HF.

But, if the given angles at the circumferences be either right or obtuse, bisect them, and the halves of them are equal; snd it can be proved, as above, that the arches upon which these halves stand are equal; whence it follows that the arches on which the given angles stand are equal.

PROP. XXVII. THEOR.

Fig. 38. *In equal circles (ABC, DEF) the angles (ABC and DEF), which stand upon equal arches, are equal, whether they be at the centres or at the circumferences.*

For, *if it be possible*, let one of them DEF be greater than the other; and make the angle DEG equal to ABC.

Because in the equal circles ABC and DEF, the
(1) Constr. angle ABC is equal to DEG (1), the arches AHC and
(2) Prop. 26. DKG are equal (2); but AHC and DKF are also
(3) Hypoth. equal (3); and therefore DKG is equal to DKF, a part equal to the whole, which is absurd: neither angle therefore is greater than the other, and therefore they are equal.

Plate 4. Fig. 54. Cor. In equal circles sectors AOC, DHF, which stand upon equal arches are equal.

Draw AC and DF, and from any points G and K draw GA, GC, KD and KF.

In the triangles AOC and DHF, the sides AO and
(1) Hypoth. OC, DH and HF are equal (1), and the angles AOC
(2) Prop. 27. and DHF are also equal (2); therefore the bases AC and DF are equal, and the triangles themselves
(3) Prop. 4. B. 1. equal (3): but if the equal arches AC and DF be taken away from the equal circles, the remaining arches ABC
(4) Ax. 3. and DEF shall be equal (4), and therefore the angles AGC and DKF also equal (2), and the segments AGC

and DKF similar (5); but the lines AC and DF are equal, therefore the segments AGC and DKF are equal (6), and therefore the sectors AOC and DHF are equal (7). (5) Def. 10. (6) Prop. 24. (7) Ax. 2.

PROP. XXVIII. THEOR.

In equal circles (ABC, DEF) equal right lines (AC and DF) cut off equal arches, the greater equal to the greater (ABC to DEF) the less to the less (AGC to DHF). Fig. 39.

If the equal right lines be diameters, the proposition is evident.

If not let K and L be the centres of the circles; and draw the lines KA, KC, LD, and LF.

Because the circles are equal (1), AK and KC are equal to LD and LF; and also AC and DF are equal (1); therefore the angle AKC is equal to the angle DLF (2), and therefore the arch AGC is equal to the arch DHF (3); and since the circles are equal, take away these equal arches from them, and the remaining arches ABC and DEF are equal. (1) Hypoth. (2) Prop. 8. B. 1. (3) Prop. 26.

PROP. XXIX. THEOR.

In equal circles (ABC, DEF) the right lines (AC and DF), which subtend equal arches, are equal. Fig. 39.

If the equal arches be semicircles, the proposition is evident.

But if not, let K and L be the centres of the circles; and draw KA, KC, LD and LF.

Because the arches AGC and DHF are equal (1), the angles AKC and DLF are equal (2); but in the triangles AKC and DLF the sides AK and KC are equal to DL and LF (3), and therefore the bases AC and DF are equal (4). (1) Hypoth. (2) Prop. 27. (3) Hypoth. & Def. 1. (4) Prop. 4. B. 1.

Schol. Whatever has been demonstrated in the preceding propositions and corollaries of equal circles, is also true of the same circle.

Cor. Hence and from propositions 26. and 27. it is evident, that in a circle right lines, which intercept equal arches, are parallel; and that parallel right lines intercept equal arches, because the alternate angles are equal.

PROP. XXX. PROB.

Fig. 40. *To bisect a given arch (ABC).*

Draw the right line AC; bisect it in E, through E draw EB perpendicular to AC; and it bisects the arch in B.

Draw the right lines AB and CB.

In the triangles AEB, CEB, the sides AE and EC
(1) Constr. are equal (1); EB is common; and the angle AEB is equal to CEB (1): therefore the sides AB and BC are
(2) Prop. 4. B. 1. equal (2), and therefore the arches, which they subtend, are equal (3); and therefore the given arch is bi-
(3) Prop.28. sected in B.

PROP. XXXI. THEOR.

Fig. 41, 42, 43. See N. *In a circle the angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is acute; and the angle in a segment less than a semicircle is obtuse.*

Fig. 41. Part 1. The angle ABC in a semicircle is a right angle.

Let O be the centre of the circle; and draw OB and AC. Because in the triangle AOB, the sides OB and OA are equal, the angles OAB and OBA are also
(1) Prop. 5. B. 1. equal (1); in the same manner it can be proved that the angles OCB and OBC are equal; therefore the angle ABC is equal to the sum of the angles BCA
(2) Cor. 1. Prop. 32. B 1. and BAC, and therefore the angle ABC is a right angle (2).

Fig. 42. Part 2. The angle ABC in a segment greater than a semicircle is acute.

Draw AD a diameter of the circle, and also the lines CD, CA. Because in the triangle ACD the angle

ACD in a semicircle is a right angle (3), the angle ADC is acute (4); but the angles ADC and ABC are in the same segment ABDC, and therefore equal (5); therefore the angle ABC is acute. (3) Part 1. (4) Cor. Prop. 17. B. 1. (5) Prop. 21.

Part 3. The angle ABC in a segment less than a semicircle is obtuse. Fig. 43.

Take in the opposite circumference any point D, and draw DA and DC.

Because in the quadrilateral figure ABCD the opposite angles B and D are equal to two right angles (6), but the angle D is less than a right angle (7), the angle ABC must be obtuse. (6) Prop. 22. (7) Part 2.

Cor. 1. Hence can be derived a method of drawing through the extremity B of any right line BC a perpendicular to it; take any point O outside the given line; describe a circle, passing through B, and cutting the line BC in any point C; draw AC, and join the points A and B by the right line AB; this line is perpendicular to CB, for the angle ABC in a semicircle is right. Fig. 41.

Cor. 2. Hence also can be drawn a tangent to a circle from a given point without it: draw a right line from the given point to the centre of the circle; bisect it, and from the point of bisection as a centre describe a circle through the given point; the right line drawn from either intersection of this circle with the given circle is a tangent: for it is perpendicular to the radius drawn to the point where it meets the circle, because the angle in a semicircle is a right angle.

PROP. XXXII. THEOR.

If a right line (EF) be a tangent to a circle, and from the point of contact a right line (AC) be drawn cutting the circle, the angle (FAC), made by this line with the tangent, is equal to the angle (ABC) in the alternate segment of the circle. Fig. 44.

If the secant should pass through the centre, it is evident that the angles are equal; for each of them is a right angle (1). (1) Prop. 18, & 31.

But if not, draw through the point of contact the line AB perpendicular to the tangent FF; and join BC.

Because the right line EF is a tangent to the circle, and AB is drawn through the point of contact perpen-
(2) Prop.19. dicular to it, AB passes through the centre (2), and
(3) Prop.31. therefore the angle ACB is a right angle (3); therefore in the triangle ABC the sum of the angles ABC and
(4) Cor. 1. Prop. 32. B. 1. BAC is equal to a right angle (4), and therefore equal to the angle BAF; take away the common angle BAC, and the remaining angle CAF is equal to the angle ABC in the alternate segment.

The angles EAC and ADC are also equal.

Draw the right lines AB and BC; because in the quadrilateral ABCD the opposite angles ABC and
(5) Prop.22. ADC taken together are equal to two right angles (5), the sum of the angles EAC and FAC is equal to the
(6) Prop.13. B. 1. sum of ABC and ADC (6); take away the equals FAC and ABC (7), and the remaining angle EAC is equal to
(7) Part pr. the angle ADC in the alternate segment.

PROP. XXXIII. PROB.

Fig. 45, 46. See N. *On a given right line (AB) to describe a segment of a circle, that shall contain an angle equal to a given angle (V).*

Fig. 45. If the given angle V be a right angle, bisect the given line in O; from the centre O, describe a circle with the radius OA; the circle is divided by the given line into two semicircles, therefore each of them contains an
(1) Prop.31. angle equal to the given right angle (1).

Fig. 46. If the given angle V be acute or obtuse, make with the given line AB, at either extremity of it A, an angle EAB or FAB equal to the given one; through A draw AC perpendicular to EF; and at B make the angle ABO equal to BAC. The circle described from the centre O, with the radius OA, passes through B, because OA
(2) Constr. & Prop. 6. B. 1. and OB are equal (2); and its segment ACB contains an angle equal to the given acute angle V, and its segment AGB contains an angle equal to the given obtuse angle V.

Because EF is a tangent to the circle at A (3), and from it is drawn AB cutting the circle, the angle in the segment ACB is equal to the angle EAB (4), and therefore to the given acute angle V (5); and also the angle in the segment AGB is equal to the angle FAB (4), and therefore to the given obtuse angle V (5). (3) Constr. & Prop. 16. (4) Prop. 32. (5) Constr.

Schol. In the same manner a circle can be described, which shall touch a given line EF at a given point, and pass through a given point B without that line.

PROP. XXXIV. PROB.

To cut off from a given circle (ABC) a segment, which shall contain an angle equal to a given angle (V). Fig. 47.

Draw FA a tangent to the circle at any point A; at the point of contact A make with the line AF an angle FAC equal to the given one V; the segment ABC contains an angle equal to the given V.

Because FA is a tangent to the circle, and AC cuts it, the angle in the segment ABC is equal to FAC (1), and therefore equal to the given angle V (2) (1) Prop. 32. (2) Constr.

PROP. XXXV. THEOR.

If two right lines (AB and CD) cut one another within a circle, the rectangle under the segments (AE and EB) of one of them is equal to the rectangle under the segments (CE and ED) of the other. Fig. 48, 49, 50.

1. If the given right lines pass through the centre, they are bisected in the point of intersection; therefore the rectangles under their segments are the squares of their halves, and therefore are equal.

2. Let one of the given lines DC pass through the centre, and the other AB not; draw OA and OB. The rectangle AEB is equal to the difference between the squares of OE and of OA (1), that is, to the difference between the squares of OE and of OC, or to the rectangle DEC (2). (1) Cor. Prop. 6. B. 2. (2) Prop. 5. B. 2.

3. Let neither of the given lines pass through the centre; draw through their intersection a diameter FG; and the rectangle under FE and EG is equal to the
(3) Part 2. rectangle under DE and EC (3), and also to the rectangle under BE and EA (3); therefore the rectangle under DE and EC is equal to the rectangle under BE
(3) Ax. 1. and EA (4).

PROP. XXXVI. THEOR.

Fig. 51, 52. *If from a point (B) without a circle two right lines be drawn to it, one of which (BF) is a tangent to the circle, and the other (BC) cuts it, the rectangle under the whole secant (BC) and the external segment (BO) is equal to the square of the tangent (BF).*

Fig. 51. 1. Let BC pass through the centre; draw AF from the centre to the point of contact; the square of BF is equal to the difference between the squares of BA
(1) Prop. 47. B. 1. and of AF (1), that is, to the difference between the
(2) Prop. 6. B. 2. squares of BA and of AO, or to the rectangle under CB and BO (2).

Fig. 52. 2. If BC does not pass through the centre, draw AO and AC. The rectangle under CB and BO is equal to the difference between the squares of AB and
(3) Cor. Prop. 6. B. 2. of AO (3), that is, to the difference between the squares of AB and AF, or to the square of BF (1).

Cor. 1. Hence, if from any point without a circle two right lines be drawn cutting the circle, the rectangles under them and their external segments are equal; for each of the rectangles is equal to the square of the tangent.

Cor. 2. If from the same point two lines be drawn to a circle, which are tangents to it, they are equal; for their squares are equal to the same rectangle.

PROP. XXXVII. THEOR.

If from a point (B) without a circle two right lines be drawn, one (BC) cutting the circle, the other (BF) meeting it, and if the rectangle under the secant and its external segment be equal to the square of the line which meets the circle, the line (BF) which meets is a tangent. Fig. 53.

Draw from the point B the line BQ a tangent to the circle, and draw EF, EQ and EB.

The square of BQ is equal to the rectangle under BC and BO (1); but the square of BF is also equal to the rectangle under BC and BO (2); therefore the squares of BF and BQ are equal, and therefore the lines themselves (3); then in the triangles EFB and EQB the sides EF and FB are equal to the sides EQ and QB, and the side EB is common, therefore the angle EFB is equal to EQB (4): but EQB is a right angle, (5); therefore EFB is a right angle, and therefore the right line BF is a tangent to the circle (6).

(1) Prop. 36.
(2) Hypoth.
(3) Cor. 2. Prop. 46. B. 1.
(4) Prop. 8. B. 1.
(5) Prop. 18.
(6) Prop. 16.

THE

ELEMENTS OF EUCLID.

BOOK IV.

DEFINITIONS.

Plate 4. Fig. 1. B. 4. See N. 1. A rectilineal figure is said to be inscribed in a circle, when the vertex of each angle of the figure is in the circumference of the circle.

Fig. 1. 2. A rectilineal figure is said to be circumscribed about a circle, when each of its sides is a tangent to the circle.

Fig. 1. 3. A circle is said to be inscribed in a rectilineal figure, when each side of the figure is a tangent to the circle.

Fig. 1. 4. A circle is said to be circumscribed about a rectilineal figure, when the circumference passes through the vertex of each angle of the figure.

Fig. 1. 5. A right line is said to be inscribed in a circle, when its extremities are in the circumference of the circle.

PROP. I. PROB.

In a given circle (BCA) to inscribe a right line, equal to a given right line (D), which is not greater than the diameter of the circle. Fig. 2.

Draw a diameter AB of the circle; and if this be equal to the given line D, the problem is solved.

If not, take in it the segment AE equal to D (1); from the centre A with the radius AE describe a circle ECF, and draw to either intersection of it with the given circle the line AC; this line is equal to AE (2), and therefore to the given line D (3).

(1) Prop. 3. B. 1.
(2) Def. 13. B. 1.
(3) Constr. & Ax. 1.

PROP. II. PROB.

In a given circle (BCA) to inscribe a triangle equiangular to a given triangle (EDF). Fig. 3.

Draw the line GH a tangent to the given circle in any point A; at the point A with the line AH make the angle HAC equal to the angle E, and at the same point with the line AG make the angle GAB equal to the angle F; and draw BC.

Because the angle E is equal to HAC (1), and HAC is equal to the angle B in the alternate segment (2), the angles E and B are equal; also the angles F and C are equal; therefore the remaining angle D is equal to BAC (3), and therefore the triangle BAC inscribed in the given circle is equiangular to the given triangle EDF.

(1) Constr.
(2) Prop. 32. B. 3.
(3) Cor. 2. Prop. 32. B. 1.

PROP. III. PROB.

About a given circle (ABC) to circumscribe a triangle equiangular to a given triangle (EDF). Fig. 4. See N.

Produce any side DF of the given triangle both ways to G and H; from the centre K of the given circle

draw any right line KA; with this line at the point K make the angle BKA equal to the angle EDG, and at the other side of KA make the angle AKC equal to EFH; and draw the lines LM, LN, and MN tangents to the circle in the points, B, A, and C.

Because the four angles of the quadrilateral figure LBKA taken together are equal to four right angles (1), and the angles KBL and KAL are right angles (2), the remaining angles AKB and ALB are together equal to two right angles; but the angles EDG and EDF are together equal to two right angles (3); therefore the angles AKB and ALB are together equal to EDG and EDF: but AKB and EDG are equal (4), and therefore ALB and EDF are equal. In the same manner it can be demonstrated that the angles ANC and EFD are equal: therefore the remaining angle M is equal to the angle E (5), and therefore the triangle LMN circumscribed about the given circle is equiangular to the given triangle.

(1) Cor. 6. Prop. 32. B. 1.
(2) Prop. 18. B. 3.
(3) Prop. 13. B. 1.
(4) Constr.
(5) Cor. 2. Prop. 32. B. 1.

PROP. IV. PROB.

Fig. 5. See N.

To inscribe a circle in a given triangle (BAC).

Bisect any two angles B and C by the right lines BD and CD, and from their point of concourse D draw DF perpendicular to any side BC; the circle described from the centre D with the radius DF is inscribed in the given triangle.

Draw DE and DG perpendicular to BA and AC. In the triangles DEB, DFB, the angles DEB and DBE are equal to the angles DFB and DBF (1), and the side DB is common to both, therefore the sides DE and DF are equal (2): in the same manner it can be demonstrated that the lines DG and DF are equal; therefore the three lines DE, DF and DG, are equal (3), and therefore the circle described from the centre D with the radius DF passes through the points E and G; and because the angles at F, E and G, are right, the lines BC, BA and AC, are tangents to the circle (4); therefore the circle FEG is inscribed in the given triangle (5).

(1) Constr.
(2) Prop. 26. B. 1.
(3) Ax. 1.
(4) Prop. 16. B. 3.
(5) Def. 3.

PROP. V. PROB.

To circumscribe a circle about a given triangle (*BAC*). Fig. 6, 7, 8.

Bisect any two sides BA and AC of the given triangle; and through the points of bisection D and E draw DF and EF perpendicular to AB and AC; and from their point of concourse F draw to any angle A of the triangle BAC the line FA; the circle described from the centre F with the radius FA is circumscribed about the given triangle.

Draw FB and FC. In the triangles FDA, FDB, the sides DA and DB are equal, FD is commmon to both, and the angles at D are right (1), therefore the sides FA and FB are equal (2): in the same manner it can be demonstrated that the lines FA and FC are equal, therefore the three lines FA, FB, and FC, are equal (3); and therefore the circle described from the centre F with the radius FA passes through B and C, and therefore is circumscribed about the given triangle BAC (4).

(1) Constr.
(2) Prop. 4. B. 1.
(3) Ax. 1.
(4) Def. 4.

Schol. This problem is the same as to describe a circle through three given points, which are not in one right line.

Cor. 1. If the centre F fall within the triangle, it is evident that all the angles are acute (1), for each of them is in a segment greater than a semicircle. If the centre F be in any side of the triangle, the angle opposite to that side is right, because it is an angle in a semicircle (1). And if the centre fall without the triangle, the angle opposite to the side which is nearest to the centre is obtuse, because it is an angle in a segment less than a semicircle (1).

Fig. 6.
(1) Prop.31. B. 3.
Fig. 7.
Fig. 8.

Cor. 2. A circle can be circumscribed about a quadrilateral figure AFDC, whose apposite angles are equal to two right angles: for draw the diagonal, and the circle described about the triangle FAC must pass through D. For take any point B in the arch FDC, and draw BF and BC. In the quadrilateral AFBC, inscribed in a circle, the angles A and B are equal to two right angles (1); but D and A are equal to two right angles (2);

Fig. 9.
(1) Prop. 22. B. 3.
(2) Hypoth.

therefore the angles B and D are equal; and because the angle B is in the circumference, the angle D must also be in the same circumference (3); therefore the circle passes through D, and therefore is circumscribed about the given quadrilateral figure AFDC.

(3) Cor. Prop. 21. B. 3.

PROP. VI. PROB.

Fig. 10. See N.

To inscribe a square in a given circle (ABCD).

Draw any diameter AC of the given circle; through the centre E draw BD perpendicular to it, and join AB, BC, CD, DA. ABCD is a square inscribed in the given circle.

Because the angles at E are right and therefore equal, the arches on which they stand are equal (1), and therefore their subtenses are equal (1); the figure ABCD is therefore equilateral: and because BD is a diameter, the angle BAD is in a semicircle and therefore right (2): in the same manner it can be demonstrated that the angles B, C and D, are right, therefore, since the sides are also equal, the figure ABCD is a square.

(1) Schol. Prop. 29. B. 3.
(2) Prop. 31. B. 3.

PROP. VII. PROB.

Fig. 11. See N.

To circumscribe a square about a given circle (ABCD).

Draw any diameter AC of the given circle, and through the centre E draw BD perpendicular to it, and through their extremities A, B, C and D, draw the lines KF, FG, GH and HK, tangents to the circle; the figure FGHK is a square circumscribed about the given circle.

Because EA is drawn from the centre to the point of contact, the angle EAF is right (1), but the angle AEB is also right (2), therefore the lines FK and BD are parallel (3): in the same manner it can be demonstrated that GH is parallel to BD, and also that FG and KH are parallel to AC; therefore GD, BK, FC and AH are parallelograms, and because the angles at A are right, the angles at G and H opposite to them are right (4): in the same manner it can be demonstrated that K and F are right angles, therefore FGHK is a

(1) Prop. 18. B. 3.
(2) Constr.
(3) Prop. 28. B. 1.
(4) Prop. 34. B. 1.

rectangle, and because AC and BD are equal, and FK and GH are each equal to BD, and FG and KH are each equal to AC (4), it is evident that FGHK is also equilateral, and therefore a square. (4) Prop. 34, B. 1.

Cor. The square circumscribed is double of the square inscribed in the same circle, for it is equal to the square of the diameter of the circle; but the square of the diameter is double of the inscribed square, as is evident from prop. 6. and from prop. 47. B. 1.

PROP. VIII. PROB.

To inscribe a circle in a given square (FGHK). Fig. 11.

Bisect two adjacent sides GH and GF of the given square in C and B, through C draw CA paralled to either FG or KH, and through B draw BD parallel to either GH or FK; the circle described from the centre E with the radius EC is inscribed in the given square.

Because GE, EH, EK and EF, are parallelograms (1), their opposite sides are equal (2), therefore CE and EB are equal to GB and GC, but GB and GC are equal, for they are halves (1) of the equal lines FG and GH, therefore CE and EB are equal; but ED and EA are equal to CE and EB, for they are equal to CH and BF the halves of GH and GF (2), therefore the four lines EC, EB, EA and ED, are equal; and therefore the circle described from the centre E with the radius EC passes through B, A and D, and because the angles at C, B, A and D, are right, the sides of the square are tangents to the circle (3); which is therefore inscribed in the given square. (1) Constr. (2) Prop. 34. B. 1. (3) Prop. 16. B. 3.

PROP. IX. PROB.

To circumscribe a circle about a given square (ABCD). Fig. 10.

Draw AC and BD intersecting one another in E; the circle described from the centre E with the radius EA must pass through B, C and D.

For, since ABC is an isosceles triangle, and the angle B is right, the other angles are each half a right angle (1); in the same manner it can be demonstrated that each of the angles, into which the angles of the square are divided, is half a right angle; they are therefore all equal; and therefore in the triangle AEB, as the angles EAB and EBA are equal, the sides EA and EB are equal (2): in the same manner it can be demonstrated that ED and EC are equal to EA and EB, therefore the four lines EA, EB, EC and ED, are equal, and therefore the circle described from the centre E, with the radius EA, passes through B, C and D, and is circumscribed about the given square.

(1) Cor. 3, Prop. 32. B. 1.
(2) Prop. 6. B. 1.

PROP. X. PROB.

Fig. 12.

To construct an isosceles triangle, in which each of the angles at the base shall be double of the vertical angle.

Take any line AB, and divide it in C, so that the rectangle under AB and CB shall be equal to the square of AC (1), from the centre A with the radius AB describe a circle BED, and inscribe in it a line BD equal to AC (2); join AD, and BAD is an isosceles triangle in which the angles B and D are each double of the angle A.

(1) Prop. 11. B. 2.
(2) Prop. 1.

Draw DC, and circumscribe a circle ACD about the triangle DCA.

Because the rectangle under AB and BC is equal to the square of AC (3), or to the square of BD (3), the line BD is a tangent to the circle ACD (4), and therefore the angle BDC is equal to the angle A in the alternate segment (5); add to both the angle CDA, and BDA is equal to the sum of the angles CDA and A; but, since the sides AB and AD are equal, the angles B and BDA are equal, therefore the angle B is equal to the sum of CDA and A; but the external angle BCD is equal to the sum of CDA and A (6), therefore the angles B and BCD are equal, and therefore the sides BD and CD are equal (7); but BD and CA are equal (3); therefore CD and CA are equal, and therefore the angles A and CDA are equal; but BDA is equal to the sum of

(3) Constr.
(4) Prop. 37. B. 3.
(5) Prop. 32. B. 3.
(6) Prop. 32. B. 1.
(7) Prop. 6. B. 1.

the angles A and CDA, therefore it is double of A, and therefore the angle B is also double of A.

Schol. It is evident that the triangle BDC is isoceles, and that each of the angles at the base is double of the angle BDC at the vertex; also that the triangle ACD is isosceles, and that the angle at the vertex ACD is triple of each of the angles at the base; for it is equal to the sum of the angles B and BDC (1), of which one is double of the angle A, the other equal to it. (1) Prop.32. B.3.

Cor. Hence is derived the method of constructing upon a given base BD an isosceles triangle, in which the angle at the base shall be double the angle at the vertex; construct a triangle of this kind CDB, in which the given line shall be a side; produce BC, and make the angle BDA equal to the angle B; BA and DA must meet, and the triangle required is constructed.

PROP. XI. PROB.

To inscribe an equilateral and equiangular pentagon in a given circle (ABCDE). Fig. 13. See N.

Construct an isosceles triangle, in which each of the angles at the base shall be double of the angle at the vertex (1); and inscribe in the given circle a triangle equiangular to it ACE (2); bisect the angles at the base A, and E, by the right lines AD and EB; and join AB, BC, CD, DE and EA. (1) Prop.10. (2) Prop. 2.

Because each of the angles CAE and CEA is double of ECA (3), and is bisected, the five angles CEB, BEA, ACE, CAD and DAE, are equal, and therefore the arches upon which they stand are equal (4), and therefore the lines CB, BA, AE, ED and DC, which subtend these arches are equal (4); and therefore the pentagon ABCDE is equilateral. (3) Constr. (4) Schol. Prop. 29. B. 3.

And because the arches AB and DE are equal, if the arch BCD be added to both, the arch ABCD is equal to BCDE (5), and therefore the angles AED and BAE standing upon them are equal (4): in the same manner it can be demonstrated that all the other angles are equal, and therefore the pentagon is also equiangular. (5) Ax. 2.

Cor. 1. Hence it is evident that every equilateral figure inscribed in a circle is equiangular.

Cor. 2. An equiangular figure, inscribed in a circle, has its alternate sides equal, as is evident from the equality of the arches which contain equal angles.

Schol. Hence if the number of sides be odd, the figure is equilateral.

Fig. 13. Cor. 3. An equilateral pentagon can be constructed upon a given line, by constructing upon this line as a base an isosceles triangle ACE, in which each of the angles at the base is double of the angle at the vertex (1), and circumscribing a circle about it; the pentagon inscribed in this circle is constructed upon the given line.

(1) Prop. 10.

PROP. XII. PROB.

Fig. 14. *To circumscribe an equilateral and equiangular pentagon about a given circle (ABCDE).*

Let the points A, B, C, D and E, be the vertices of the angles of an equilateral pentagon inscribed in the circle, and draw GH, HK, KL, LM and MG, tangents to the circle at these points; GHKLM is an equilateral and equiangular pentagon, circumscribed about the given circle.

Draw FA, FG, FE, FM and FD. In the triangles FGA, FGE, the sides GA and GE are equal (1), and also FA and FE, FG is common, therefore the angles FGA and FGE are equal, and also AFG and EFG (2); therefore the angle AGE is double of FGE, and AFE double of GFE; in the same manner it can be demonstrated that DME is double of FME, and that DFE is double of MFE; but, since the arches AE and ED are equal (3), the angles AFE and DFE are equal (4), and therefore their halves GFE and MFE are equal, and the angles FEG and FEM are also equal, and the side EF is common, therefore the angles FGE and FME are equal, and also the sides GE and EM (5), and therefore the line GM is double of GE: in the same manner it can be demonstrated that GH is double of GA; but GE and GA are equal, therefore GM and

(1) Cor. 2. Prop. 36. B. 3.
(2) Schol. Prop. 8. B. 1.
(3) Constr.
(4) Schol. Prop. 29. B. 3.
(5) Prop. 26. B. 1.

GH are equal: in the same manner it can be demonstrated that the other sides are equal, and therefore the pentagon GHKLM is equilateral: and because the angles DME and AGE are double of FME and FGE, and FME and FGE are equal, DME is equal to AGE (6); and in the same manner it can be demonstrated (6) Ax. 6.
that the other angles are equal, and therefore GHKLM is also equiangular.

Cor. In the same manner any equilateral and equiangular figure can be circumscribed about a circle; by inscribing a similar figure, and drawing tangents through the vertices of its angles.

PROP. XIII. PROB.

To inscribe a circle in a given equilateral and equiangular pentagon (ABCDE). Fig. 15.

Bisect any two adjacent angles A and E by the right lines AF and EF, and from their point of concourse F draw FG perpendicular to AE; the circle described from the centre F with the radius FG is inscribed in the given pentagon.

Draw FB, FC and FD; and from F let fall the perpendiculars FH, FN, FM, FL.

In the triangles AFB, AFE, the sides AB and AE are equal (1), AF is common, and the angles FAB and (1) Hypoth.
FAE are equal (2), therefore the angles ABF and (2) Constr.
AEF are equal (3); but the angles ABC and AED are (3) Prop. 4.
also equal (1), therefore since AEF is half of AED (2), B. 1.
ABF is half of ABC: in the same manner it can be demonstrated that the other angles of the pentagon are bisected by the lines drawn from F: wherefore in the triangles FBH, FBN, the angles FBH and FBN are equal, the angles at H and N are right, and the side FB opposite to the equal angles H and N is common, therefore the sides FH and FN are equal (4); and in the same manner it is proved that all the perpendicu- (4) Prop. 26.
lars are equal: therefore the circle described from the B. 1.
centre F, with the radius FG, passes through the points H, N, M and L, and the sides of the given pentagon are tangents to it, because the angles at G, H, N, M and L, are right.

Schol. In the same manner a circle can be inscribed in any equilateral and equiangular figure.

PROP. XIV. PROB.

Fig. 16. *To circumscribe a circle about a given equilateral and equiangular pentagon (ABCDE).*

Bisect the angles A and E by the right lines AF and EF; the circle described from the point of concourse F as centre, with the radius AF, passes through the points B, C, D and E.

Draw FB, FC and FD. In the triangles FAE and FAB, the sides FA and AE are equal to FA and AB,
(1) Constr. and the angle FAE is equal to FAB (1), therefore the
(2) Prop. 4. B. 1. angles FBA and FEA are equal (2), but the angles ABC and AED are also equal (3), therefore since the
(3) Hypoth. angle FEA is half of AED (1), FBA is half of ABC, and therefore ABC is bisected by FB; and in the same manner it can be demonstrated that the angles C and D are bisected: Hence in the triangle AFE the angles FAE and FEA, being halves of the equal angles BAE and AED, are equal, and therefore the sides FE and
(4) Prop. 6. B. 1. FA are equal (4); and in the same manner it is proved that the remaining lines FB, FC and FD, are equal; therefore the five lines FA, FB, FC, FD and FE, are equal, and therefore the circle described from the centre F, with the radius FA, passes through the points B, C, D and E, and therefore is circumscribed about the given pentagon.

Schol. In the same manner a circle can be circumscribed about any equilateral and equiangular figure.

PROP. XV. PROB.

Fig. 17. *To inscribe an equilateral and equiangular hexagon in a given circle (ABCDEF).*

Let G be the centre of the given circle, draw any diameter AGD; from the centre A with the radius AG describe a circle, and from its intersections B and F with

the given circle draw the diameters BE and FC, join AB, BC, CD, DE, EF and FA; and the figure ABCDEF is an equilateral and equiangular hexagon inscribed in the given circle.

Since the lines AB and AG are equal, as being radii of the same circle BGF, and GA and GB also equal, as being radii of the same circle ABCDEF, the triangle BGA is equilateral, and therefore the angle BGA is the third part of two right angles (1); in like manner it is proved that the triangle AGF is equilateral, and the angle AGF equal to one third part of two right angles; but the angles BGA and AGF together with FGE are equal to two right angles (2), therefore FGE is one third part of two right angles, and therefore the three angles BGA, AGF and FGE, are equal, and also the angles vertically opposite to them EGD, DGC and CGB; hence the six angles at the centre G are equal, therefore the arches on which they stand are equal, and the lines subtending those arches (3), therefore the hexagon ABCDEF is equilateral, and also, since it is inscribed in a circle, equiangular (4).

(1) Cor. 4. Prop. 32. B. 1.
(2) Cor. 1. Prop. 13. B. 1.
(3) Schol. Prop. 29. B. 3.
(4) Cor. 1. Prop. 11.

Cor. 1. The side of a hexagon is equal to the radius of the circle, in which it is inscribed.

Cor. 2. If the arches AB, BC, &c. were bisected, a figure of twelve sides might be inscribed, and these arches being bisected, a figure of twenty-four sides, and so on. Fig. 17.

Cor. 3. An equilateral and equiangular hexagon can be constructed upon a given line BA, by describing upon it an equilateral triangle BGA, and from the centre G with the radius GA describing a circle, and inscribing in it a hexagon. Fig. 17.

Cor. 4. Draw AC, AE and CE, and an equilateral triangle is inscribed in the circle, whose sides bisect the radii perpendicular to them. For let GK be perpendicular to CE, it passes through D; therefore in the triangles DCK, GCK, the angles at K are right, the angles DCK and GCK, standing upon the equal arches DE and EF, are also equal (1), and the side CK between the equal angles common, therefore the sides DK and KG are equal (2); and therefore GD is bisected in K. Fig. 17.

(1) Schol. Prop. 22. B. 3.
(2) Prop. 26. B. 1.

Cor. 5. Hence it follows that the square of the side of an equilateral triangle is triple the square of the ra-

dius of the circle in which it is inscribed; draw AC, and its square is equal to the sum of the squares of AG and GC and to twice the rectangle under AG and GK (1), but KG is the half of GD (2), therefore double the rectangle AGK is equal to the square of AG; therefore the square of AC is equal to twice the square of AG and the square of GC, or to triple the square of radius.

(1) Prop. 12. B. 2.
(2) Cor. pr.

PROP. XVI. PROB.

Fig. 18.
See N.

To inscribe an equilateral and equiangular quindecagon in a given circle (CAD).

Let CD be the side of an equilateral triangle inscribed in the circle CAD, and CA the side of an equilateral pentagon also inscribed in the circle CAD; bisect the arch AD; the right line joining AB is the side of the required quindecagon. For if the whole circumference be divided into fifteen parts, the arch CD, since it is the third part of the whole circumference, contains five of these parts; in like manner the arch CA contains three of them, therefore the arch AD contains two; and therefore the arch AB is the fifteenth part of the whole circumference, and AB is the side of the required equilateral quindecagon.

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BOOK V.

DEFINITIONS.

1. A less magnitude is called an aliquot part or submultiple of a greater, when the less measures the greater.

2. A greater magnitude is called a multiple of a less, when the greater is measured by the less.

3. Ratio is a mutual relation of two magnitudes of the same kind, with respect to quantity.

4. Magnitudes are said to have a ratio to each other, if they be such that the less can be multiplied so as to exceed the greater. See N.

5. Magnitudes are said to be in the same ratio, the first to the second as the third to the fourth, when any submultiple whatsoever of the first is contained in the second, as often as an equi-submultiple of the third is contained in the fourth. See N.

6. Magnitudes which have the same ratio are called proportionals.

7. If a submultiple of the first be contained in the second oftener than an equi-submultiple of the third is contained in the fourth; the first is said to have a less ratio to the second than the third has to the fourth: and *e contra*, the third is said to have a greater ratio to the fourth than the first has to the second. See N.

8. Proportion is the similitude of ratios.

9. Proportion consists of three terms at the least.

10. When three magnitudes are proportional (A to B as B to C) the first is said to have to the third (A to C) a duplicate ratio of that which it has to the second (that is, of the ratio of A to B.)

11. When four magnitudes are in continued proportion (A to B as B to C, and B to C as C to D) the first is said to have to the fourth (A to D) a triplicate ratio of that which it has to the second (that is, of the ratio of A to B).

See N. 12. If there be any number of magnitudes of the same kind (A, D, C and F) the first is said to have to the last (A to F) a ratio compounded of the ratios which the first has to the second, the second to the third, the third to the fourth (A to D, D to C, and C to F); and so on to the last.

13. In proportionals the antecedents are said to be homologous to one another, as also the consequents to one another.

Geometers make use of the following terms to express certain modes of changing either the order or magnitude of proportionals, so that they continue still to be proportionals.

See N. 14. By permutation or alternation; when we infer that if there be four magnitudes of the same kind proportional, the first is to the third as the second to the fourth, *as is proved in prop.* 33.

15. By inversion; when we infer that if there be four magnitudes proportional, the second is to the first as the fourth to the third. *Prop.* 20.

16. By composition; when we infer that if there be four magnitudes proportional, the sum of the first and second is to the second, as the sum of the third and fourth to the fourth. *Cor. prop.* 21.

17. By division; when we infer that if there be four magnitudes proportional, the difference between the first and second is to the second, as the difference between the third and fourth to the fourth. *Cor.* 2. *prop.* 25.

18. By conversion; when we infer that if there be four magnitudes proportional, the first is to the sum or difference of the first and second, as the third to the

sum or difference of the third and fourth. *Prop.* 21, and 25. and *Cor.* 1. *prop.* 25.

19. Ex æquali, or ex æquo; when we infer that if there be any number of magnitudes more than two, and as many others, which taken two by two of each series are in the same ratio, the first is to the last in the first series, as the first to the last in the second series.

Of this there are two species.

20. Ex æquo ordinate; when the first magnitude is to the second in the first series, as the first to the second in the second series; and the second to the third in the first series, as the second to the third in the second series; and so on; and we infer, as in Def. 19. that the first is to the last in the one series as the first to the last in the other. *Prop.* 34.

21. Ex æquo perturbate; when the first magnitude is to the second in the first series, as the penultimate to the last in the second series; and the second to the third in the first series, as the antepenultimate to the penultimate in the second series; and so on; and we infer, as in Def. 19. that the first is to the last in the one series as the first to the last in the other. *Prop.* 38.

Axioms.

1. Equi multiples or equi-submultiples of the same or of equal magnitudes are equal. See N.

2. Those magnitudes, of which the same or equal magnitudes are equi-multiples or equi-submultiples, are equal.

3. A multiple or submultiple of a greater magnitude is greater than the equi-multiple or equi-submultiple of a less.

4. That magnitude, of which a multiple or submultiple is greater than the equi-multiple or equi-submultiple of another, is greater than that other magnitude.

PROP. I. THEOR.

Plate 5. Fig. 1. See N.

If there be given two equal magnitudes (BC and DE), as often as any third magnitude (A) is contained in one of them, so often it is contained in the other.

1. Let one of the given magnitudes BC be a multiple of A, and A is not oftener contained in one of them than in the other.

For, *if it be possible,* let A be oftener contained in DE, than in BC; and as often as A is contained in BC, so often take it away from DE, and let a certain part mE remain; BC and Dm are therefore equi-multiples of the same A, and therefore BC is equal to Dm (1); but BC is equal to DE (2), therefore Dm is equal to DE (3), which is absurd.

(1) Ax. 1. (2) Hypoth. (3) Ax. 1. B. 1.

Now, *if it be possible,* let A be oftener contained in BC than in DE; take away A as often as it is possible from DE, and a part mE shall remain less than A; take it away as often from BC, and since it is contained in BC oftener than in DE, a part nC shall remain greater than A or equal to it, and therefore greater than mE: but Bn and Dm are equi-multiples of A (4), and therefore equal (1), also BC and DE are equal (2), therefore nC is equal to mE (5); but it was proved to be greater than it, which is absurd.

(4) Constr. (5) Ax. 3. B. 1.

2. Let neither of the given magnitudes be a multiple of A; and A is not oftener contained in one than in the other.

For, *if it be possible,* let A be oftener contained in BC than in DE; take it away from DE as often as it is possible, and mE shall remain less than A; take away A as often from BC, and nC shall remain greater than A, and therefore greater than mE; but Bn and Dm are equi-multiples of A and therefore equal (1), also BC and DE are equal (2), therefore nC is equal to mE (5); but it has been proved greater than it, which is absurd. In no case therefore is A oftener contained in one of the given magnitudes, BC or DE, than in the other.

Cor. 1. Hence it is evident, that if either of the given magnitudes be a multiple of A, the other is also an equi-multiple of A.

Cor. 2. If two magnitudes be equal, as often as one of them is contained in any third, so often is the other contained in the same third.

Cor. 3. If there be four magnitudes, the first equal to the second, the third equal to the fourth, as often as the first is contained in the third, so often the second is contained in the fourth.

PROP. II. THEOR.

If there be given two magnitudes (BC and DE), and any submultiple whatsoever of a third magnitude (A) be always contained as often in the one as in the other, those given magnitudes (BC and DE) are equal. Fig. 2.

For, *if it be possible*, let one of them BC be greater than the other, and let its excess be nC; take a submultiple *a* of A less than nC, *a* is oftener contained in BC than in Bn; but Bn is equal to DE, therefore *a* is contained in Bn as often as in DE (1), and therefore is contained in BC oftener than in DE; but it is contained as often (2), which is absurd; therefore neither of the given quantities is greater than the other, and therefore they are equal.

(1) Prop. 1.
(2) Hypoth.

Cor. If there be four magnitudes, of which the first is equal to the second, and any submultiple of the first be always contained in the third as often as an equi-submultiple of the second is contained in the fourth, the third is equal to the fourth.

PROP. III. THEOR.

If there be four magnitudes (A, B, CD and EF), of which the first is greater than the second, and the third is equal to the fourth, the first is not contained in the third oftener than the second in the fourth. Fig. 3.

1. Let B be a submultiple of EF; and, *if it be possible*, let A be contained in CD oftener than B in EF;

let A be taken away from CD as often as B is contained in EF, and let mD be the remainder; because Cm (1) Constr. and EF are equi-multiples of A and B (1), and A is (2) Hypoth. greater than B (2), Cm is greater than EF (3); but EF (3) Ax. 3. is equal to CD (2), therefore Cm is greater than CD, which is absurd.

2. Let B be not a submultiple of EF; and, *if it be possible*, let A be contained in CD oftener than B is contained in EF; take away B as often as possible from EF, and there shall remain nF less than B; take away A as often from CD, and since A is contained in CD oftener than B is contained in EF, the remainder mD is either greater than A or equal to it, and therefore greater than nF; but Cm and En are equi-multiples of A and B (1), and A is greater than B, therefore Cm is greater than En (3); and also mD is greater than nF, therefore the whole CD is greater than EF, but it is also equal to it (2), which is absurd. In no case therefore is A contained in CD oftener than B is contained in EF.

Cor. In like manner, if there be four magnitudes, of which the first is equal to the second, and the third less than the fourth, the first is not contained in the third oftener than the second in the fourth.

PROP. IV. THEOR.

Fig. 4. *If there be given two magnitudes (AB and CD), of which one (AB) is greater than the other (CD), there is a submultiple of any third magnitude (E), which is contained in the greater oftener than in the less.*

Let mB be the excess of AB above CD; take *e* a submultiple of E less than mB; and since Am is equal to CD, *e* is contained in Am as often as it is contained (1) Prop. 1. in CD (1); but *e* is less than mB (2), therefore *e* is con- (2) Constr. tained in AB oftener than in Am, and therefore oftener than in CD.

PROP. V. THEOR.

If there be given two magnitudes (AB and CD), and any third can be found (e) which is contained in one (AB) oftener than in the other (CD), the latter (CD) is less than the former. Fig. 4, 5.

If e be a submultiple of CD, take it away from AB as often as it is contained in CD, and let mB be the remainder (1); Am is equal to CD because they are equimultiples of e, therefore AB is greater than CD. Fig. 4. (1) Hypoth.

But if e be not a submultiple of CD, take it away from CD as often as possible, and let the remainder be nD less than e; take it away as often from AB, and since it is contained in AB oftener than in CD (1), the remainder mB is greater than e, or equal to it, and therefore greater than nD; but Am and Cn are equimultiples of e (2), and therefore equal (3): and mB is greater than nD, therefore the whole AB is greater than CD. Fig. 5. (2) Constr. (3) Ax. 1.

PROP. VI. THEOR.

If there be given two magnitudes (a and AB), of which the former is a submultiple of the latter, and there be taken any third magnitude (x), which is a submultiple of the former (a), it is also a submultiple of the latter (AB). Fig. 6.

Divide AB into parts Ao and oB equal to a; and since x is a submultiple of a (1), it shall be a submultiple of Ao (2) and also of oB; divide Ao and oB into parts Am, mo, on, nB, equal to x; and the whole AB is divided into parts equal to x, and therefore x is a submultiple of AB. (1) Hypoth. (2) Cor. Prop. 1.

PROP. VII. THEOR.

If there be two magnitudes (a and b) equi-submultiples of two other magnitudes (AC and BD), and there be taken two others (x and z) which are equi-submultiples of the former, they are also equi-submultiples of the latter. Fig. 7.

Divide AC into parts Ao, oC, equal to a; and BD into parts Bn, nD, equal to b; because a and Ao are

(1) Constr. (2) Hypoth. (3) Cor. 1. Prop. 1. equal (1), and x is a submultiple of a (2), it must be an equi-submultiple of Ao (3); in the same manner it is proved that z is an equi-submultiple of b and Bn; but x and z are equi-submultiples of a and b (2), and therefore of Ao and Bn, and therefore also equi-submultiples of oC and nD; whatever submultiple therefore x is of AC, the same is z of BD.

Fig. 8. See N. Cor. 1. If z be a submultiple of b, and be contained in b as often as x is contained in an, and there be taken equi-multiples BD and AC of b and an, z is not contained in BD oftener than x in AC.

(1) Prop. 7. For if x be a submultiple of an, z and x are equi-submultiples of BD and AC (1), and therefore x is contained in AC as often as z is contained in BD.

(2) Hypoth. (3) Ax. 3. (4) Cor. Prop. 3. But if x be not a submultiple of an, take it away from an as often as possible, and let the remainder be on; since x and z are equi-submultiples of ao and b (2), as often as z is contained in any multiple of b, so often is x contained in an equi-multiple of ao; but an is greater than ao, and therefore any multiple of an is greater than the equi multiple of ao (3); therefore x is not contained in a multiple of ao oftener than in the equi-multiple of an (4); and therefore z is not contained in a multiple of b oftener than x in the equi-multiple of an.

Fig. 7, 9. Cor. 2. Let b be a submultiple of BD, and be contained in BD as often as a is contained in AC, and let z and x be equi-submultiples of b and a; z is not contained in BD oftener than x in AC.

(1) Prop. 7. If a be a submultiple of AC, it is evident that x and z are equi-submultiples of AC and BD (1).

Fig. 9. (2) Hypoth. But if not, take it away from AC as often as possible, and let the remainder be oC; a and b are equi-submultiples of Ao and BD, and therefore, since x and z are equi-submultiples of a and b (2), as often as z is contained in BD, so often is x contained in Ao (1), and therefore z is not contained in BD oftener than x in AC.

Fig. 10. Cor. 3. Let z be a submultiple of BD and be contained in it, as often as x is contained in the same BD; x is either equal to z or less than it.

Let x be a submultiple of BD; and it is evident that x and z are equal, because they are equi-submultiples of the same BD (1). (1) Ax. 1.

But if not, take away x from BD as often as z is contained in it, and let the remainder be mD; x and z are equi-submultiples of Bm and BD, and Bm is less than BD, therefore x is less than z (2). (2) Ax. 3.

PROP. VIII. THEOR.

If there be two unequal magnitudes (A and B), there is a submultiple of the less (B), which is contained in a third magnitude (CD) oftener than an equi-submultiple of the greater (A). Fig. 11, 12.

Take any submultiple a of A, and b an equi-submultiple of B.

1. Let a be a submultiple of CD; take away b from CD, as often as a is contained in CD; and because b is less than a (1), there must be a remainder mD: if mD is greater than b, b is contained in CD oftener than in Cm, and therefore oftener than a is contained in CD. Fig. 11. (1) Hypoth. & Ax. 3.

But if mD be less than b, take z a submultiple of b less than mD, and x an equi-submultiple of a; since x and z are equi-submultiples of a and b (2), and a and b equi-submultiples of CD and Cm, x and z are equi-submultiples of CD and Cm (3), but z is less than mD (2), therefore z is contained in CD oftener than in Cm, and therefore oftener than x is contained in the same CD. (2) Constr. (3) Prop. 7.

2. Let a not be a submultiple of CD; take it away as often as possible from CD, and let the remainder be nD; take away b as often from CD, and let the remainder be mD; since Cm and Cn are equi-multiples of b and a, but b is less than a (1), Cm is less than Cn (4): take z a submultiple of b less than mn, and x an equi-submultiple of a; z and x are equi-submultiples of Cm and Cn (3): but z is less than mn (2), therefore z is contained in Cn oftener than in Cm, and therefore oftener than x is contained in Cn; but x is greater than z (5), therefore x is not contained in nD oftener than z Fig. 12. (4) Ax. 3. (5) Constr. & Ax. 3.

(6) Prop. 3. in the same nD (6); but z is oftener contained in Cn than x in Cn, and therefore z is contained in CD oftener than x in CD.

Cor. In like manner, if there be four magnitudes, of which the first A is greater than the second C, but the third B equal to the fourth D; there can be taken a submultiple of C, which is contained in D oftener than an equi-submultiple of A is contained in B.

PROP. IX. THEOR.

Fig. 13. *If there be given two magnitudes (B and A), and there be given a submultiple (b) of one of them, contained in any third magnitude (CD) oftener than an equi-submultiple (a) of the other is contained in the same third, the latter (A) is greater than the former (B).*

1. Let b be a submultiple of CD, add a to itself as often as b is contained in CD, and let the sum be EF; a is contained in EF as often as b in CD, but b is contained in CD oftener than a in CD (1), therefore a is contained in EF oftener than in CD, and therefore EF is greater than CD (2); but a and b are equi-submultiples of EF and CD (3), therefore a is greater than b (4); and therefore, since A and B are equi-multiples of a and b, A is greater than B (4).

(1) Hypoth.
(2) Prop. 5.
(3) Constr. & Hypoth.
(4) Ax. 3.

2. Let b not be a submultiple of CD; take it away as often as possible from CD, and let the remainder be mD; make EF as before, and it shall be greater than CD, and therefore than Cm; but a and b are equi-submultiples of EF and Cm, therefore a is greater than b; and therefore, since A and B are equi-multiples of a and b, A is greater than B.

PROP. X. THEOR.

If there be four magnitudes (x, z, a and b), of which the first is contained in the third oftener than the second in the fourth, the first is contained in any multiple (AC) of the third oftener than the second is contained in the equi-multiple (BD) of the fourth. Fig. 14.

Divide AC into parts Am and mC equal to a, and BD into Bn and nD equal to b; since x is contained in a oftener than z in b (1), and x is contained in Am as often as in a, and z as often in Bn as in b (2), x is contained in Am oftener than z in Bn; take away z from BD, if it be possible, as often as x is contained in Am and the remainder oD is less than nD, therefore x is contained in Am as often as z in Bo (3); but x is contained in mC oftener than z in nD (4), therefore oftener than z in oD (5), and therefore x is contained in AC oftener than z in BD.

(1) Hypoth. (2) Prop. 1. (3) Constr. (4) Hypoth. & Prop. 1. (5) Cor. Prop. 3.

But if z cannot be taken away from BD as often as x is contained in Am, it is evident that x is contained in AC oftener than z in BD.

Cor. Hence if x be contained in a oftener than z in b, and a be contained in AC oftener than b in BD, x is contained in AC oftener than z in BD.

PROP. XI. THEOR.

If two equal magnitudes (AC and BD) be divided into aliquot parts unequal in number (Am and mC; Bn, no and oD), the part (Bn) of those which are greater in number is less than the part (Am) of those which are less in number. Fig. 15.

As often as Bn is contained in BD, so often add Am to itself, and let the sum be EF; since Am is contained in EF as often as Bn in BD (1), but Bn is contained in in BD oftener than Am in AC (2), Am is contained in EF oftener than in AC; therefore EF is greater than AC (3), and therefore greater than BD; but Am and Bn are equi-submultiples of EF and BD (1), and therefore Bn is less than Am (4).

(1) Constr. (2) Hypoth. (3) Prop. 5. (4) Ax. 3.

PROP. XII. THEOR.

Fig. 16. *If there be any number of magnitudes (a and b), which are equi-submultiples of other magnitudes (A and B) equal in number, whatever submultiple one is of its multiple, the same are all the submultiples taken together of all the multiples taken together.*

Since *a* is contained in AC as often as *b* is contained (1) Hypoth. in BD (1), there are as many parts in AC equal to *a*, as there are in BD equal to *b*: divide AC into the parts Am, mC, equal to *a*; and BD into Bn, nD, equal to *b*; the number of the parts Am, mC, must be equal to the number of parts Bn, nD; and, since Am is equal to *a*, and Bn to *b*, Am and Bn together are equal to *a* and *b*; likewise mC and nD together are equal to *a* and *b*: therefore there are as many parts in AC and BD together equal to *a* and *b* taken together, as there are parts in AC equal to *a*; and therefore whatever submultiple *a* is of AC, the same submultiple is *a* together with *b* of AC and BD taken together.

Fig. 16, 17. Cor. 1. If there be four magnitudes (*a*, *b*, AC and BD), of which the first is contained in the third as often as the second in the fourth, the first together with the second shall be contained as often in the third together with the fourth.

Fig. 16. (1) Prop. 12. If *a* and *b* be submultiples of AC and BD, the proposition is evident (1).

Fig. 17. But if neither be a submultiple, take away *a* from AC as often as possible, and let the remainder be mC less than *a*; take away *b* from BD as often, and let the remainder be nD less than *b*; *a* and *b* are equi-submultiples of Am (2) Hypoth. & Constr. and Bn (2), and therefore as often as *a* is contained in Am, so often *a* together with *b* is contained in Am together with Bn (1); but *a* is greater than mC, and *b* is greater than nD, therefore *a* and *b* together are greater than mC and nD, and are not contained in AC and BD taken together oftener than in Am and Bn taken together; and therefore *a* together with *b* is contained in

AC and BD taken together as often as a is contained in AC.

In the same manner it can be demonstrated, if a be a submultiple of AC, and b not a submultiple of BD.

Cor. 2. This is also true of any number of magnitudes.

PROP. XIII. THEOR.

If there be four magnitudes proportional, the first to the second as the third to the fourth (A to CD as B to EF), and a submultiple (a) of the first be a submultiple of the second, an equi-submultiple (b) of the third is a submultiple of the fourth. Fig. 18.

For, *if it be possible,* let a be a submultiple of CD, and b not a submultiple of EF; take away b from EF as often as a is contained in CD, and let the remainder be oF; take z a submultiple of b less than oF, and x an equi-submultiple of a; since x and z are equi-submultiples of a and b, and a and b equi-submultiples of CD and Eo, as often as x is contained in CD, so often z is contained in Eo (1), but oF is greater than z (2), and therefore z is contained oftener in EF than in Eo, and therefore z is contained in EF oftener than x in CD; but since x and z are equi-submultiples of a and b (2), they are equi-submultiples of A and B (1), and therefore x is contained in CD as often as z in EF (3); but it was proved before that z was contained in EF oftener than x in CD, which is absurd: therefore there can be no remainder when b is taken as often as possible from EF, and therefore b is a submultiple of EF. (1) Prop. 7. (2) Constr. (3) Hypoth. & Def. 5.

PROP. XIV. THEOR.

Equal magnitudes (A and B) have the same ratio to the same magnitude (C). Fig. 19.

And the same magnitude (C) has the same ratio to equal magnitudes (A and B).

Part 1. Take any equi-submultiples a and b of the equal magnitudes A and B, these submultiples are

(1) Ax. 1. equal (1); therefore as often as a is contained in any
(2) Cor. 2. third magnitude C, so often b is contained in the same
Prop. 1. (2), and therefore A is to C as B to C (3).

(3) Def. 5. Part 2. Take any submultiple c of C; since A and
(4) Hypoth. B are equal (4), c is contained in A as often as it is in B
(5) Prop. 1. (5); therefore C is to B as C to A (3).

Cor. If there be four magnitudes A, B, C and D, of which the first is equal to the second, and the third equal to the fourth, A is to C as B to D.

PROP. XV. THEOR.

Fig. 19. *Magnitudes (A and B), which have the same ratio to the same magnitude (C), are equal.*

And magnitudes, to which the same magnitude (C) has the same ratio, are equal.

Part 1. For, *if it be possible,* let one of them A be less than the other, and there can be taken a submultiple of it which is contained in C oftener than an equi-
(1) Prop. 8. submultiple of the greater B is contained in C (1), con-
(2) Def. 5. trary to the hypothesis (2); therefore neither of them is greater than the other, and therefore they are equal.

Part 2. For, *if it be possible,* let one of them B be greater than the other, and a submultiple of C can be
(3) Prop. 4. taken, which is contained in B oftener than in A (3), contrary to the hypothesis (2); therefore neither of them is greater than the other, and therefore they are equal.

PROP. XVI. THEOR.

Fig. 20. *Of unequal magnitudes (A and B) the less (A) has to any magnitude (C) a less ratio than the greater.*

And any magnitude (C) has to the less (A) a greater ratio than to the greater (B).

Part I. Since A is less than B, a submultiple of A can be taken, which is contained in C oftener than an equi-
(1) Prop. 8. submultiple of B is contained in the same C (1); there-
(2) Def. 7. fore A has to C a less ratio than B has to the same C (2).

Part 2. Since A is less than B, a submultiple of C can be taken, which is contained in B oftener than in A (3); and therefore C has a greater ratio to A than to B (2). (3) Prop. 7. (2) Def. 7.

PROP. XVII. THEOR.

Of two magnitudes (A and B) that (B), which has the greater ratio to any magnitude (C), is greater than the other. Fig. 20.

And that magnitude (A) to which the same (C) has the greater ratio, is less than the other (B).

Part 1. Since the ratio which B has to C is greater than the ratio of A to C, a submultiple of A can be taken, which is contained in C oftener than an equi-submultiple of B is contained in the same C (1); and therefore A is less than B (2). (1) Def. 7. (2) Prop. 9.

Part 2. Since C has a greater ratio to A than to B, a submultiple of C can be taken, which is contained in B oftener than in A (1); and therefore B is greater than A (3). (3) Prop. 5.

PROP. XVIII. THEOR.

Those ratios, which are the same with any third ratio, are the same. Fig. 21.

If A is to B as C to D, and C to D as E to F, A is to B as E to F.

Since A is to B as C to D, as often as any submultiple of A is contained in B, so often an equi-submultiple of C is contained in D (1); and since C is to D as E to F, as often as any submultiple of C is contained in D, so often an equi-submultiple of E is contained in F (1); therefore as often as any submultiple of A is contained in B, so often an equi-submultiple of E is contained in F; and therefore A is to B as E to F (1). (1) Def. 5.

PROP. XIX. THEOR.

Fig. 21. *If the first magnitude has to the second the same ratio that the third has to the fourth (A to B as C to D), but the third has to the fourth a greater ratio than the fifth has to the sixth (C to D than E to F), the first has to the second a greater ratio than the fifth has to the sixth (A to B than E to F).*

Since C has to D a greater ratio than E has to F, a submultiple of E can be taken, which is contained in F oftener than an equi-submultiple of C is contained in D
(1) Def. 7. (1); and therefore, since A is to B as C to D, oftener
(2) Def. 5. than an equi-submultiple of A is contained in B (2); and therefore A has to B a greater ratio than E has to F (1).

Cor. In the same manner, if A be to B as C to D, but E to F in a greater ratio than C to D, it can be proved that E has to F a greater ratio than A has to B.

PROP. XX. THEOR.

Fig. 22. *If there be four magnitudes proportional (A to B as CD to E) they are also proportional when taken inversely (B to A as E to CD).*

If there be taken any equi-submultiples b and e of B and E, b is contained in A as often as e is contained in CD.

For, *if it be possible*, let one of them b be contained in A oftener than e is contained in CD; let e be repeated as often as b is contained in A, so as to make up
(1) Prop. 5. CY; CY is greater than CD (1), and e is a submultiple of CY; let x be an equi-submultiple of CD, and also x of A.

Because CD is less than CY, and x and e are equi-
(2) Ax. 3. submultiples of them, x is less than e (2), therefore there is a submultiple of x, which is contained in E oftener than an equi-submultiple of e is contained in the same

E (3), that is oftener than an equi-submultiple of b is contained in B (4); but since z repeated as often as b is contained in A makes up A itself (5), z is either greater than b or equal to it (6); therefore no submultiple of z is contained in B oftener than an equi-submultiple of b is contained in B (7), but there is a submultiple of x, which is contained in E oftener than an equi-submultiple of b is contained in B, and therefore oftener than an equi-submultiple of z is contained in B; but x and z are equi-submultiples of CD and A (3); therefore equi-submultiples of x and z are also equi-submultiples of CD and A (8); therefore there is no submultiple of x which is contained in E oftener than an equi-submultiple of z is contained in B (9); but there is one which is absurd: therefore there is no submultiple of B, which is contained in A oftener than an equi-submultiple of E is contained in CD: and in the same manner it can be demonstrated that there is no submultiple of E, which is contained in CD oftener than an equi-submultiple of B is contained in A: therefore B is to A, as E to CD (10).

(3) Prop. 8. (4) Hypoth. & Prop. 7, (5) Constr. (6) Cor. 3. Prop. 7. (7) Prop. 9. (8) Prop. 7. (9) Hypoth. & Def. 5. (10) Def. 5.

PROP. XXI. THEOR.

If there be four magnitudes proportional (A to B as C to D), the first is to the sum of the first and second, as the third to the sum of the third and fourth (A to sum of A and B, as C to sum of C and D). Fig. 23.

Take a and c equi-submultiples of A and C, and a is contained in B as often as c is contained in D (1): therefore, since a is contained in A as often as c in C, and in B as often as c in D, a is contained in the sum of A and B as often as c in the sum of C and D; and therefore A is to the sum of A and B, as C to the sum of C and D (2).

(1) Hypoth. & Def. 5. (2) Def. 5.

Cor. If A be to B as C to D, the sum of A and B is to B as the sum of C and D to D.

Since A is to B as C to D, B must be to A, as D to C (1), therefore B to the sum of B and A, as D to the sum of D and C (2); and therefore the sum of A and B, to B, as the sum of C and D to D (1).

(1) Prop. 20. (2) Prop. 21.

PROP. XXII. THEOR.

Fig. 23. *If there be any number of magnitudes proportional (A to B as C to D), as one of the antecedents is to its consequent (A to B), so is the sum of all the antecedents to the sum of all the consequents (sum of A and C to sum of B and D.)*

For, if there be taken *a* and *c* equi-submultiples of A and C, the sum of *a* and *c* is the same submultiple
(1) Prop. 12. of the sum of A and C (1); but *a* is contained in B as
(2) Hypoth. & Def. 5. often as *c* is contained in D (2), and therefore as often
(3) Cor. 1. Prop. 12. as the sam of *a* and *c* is contained in the sum of B and D (3); and therefore A is to B as the sum of A and C
(4) Def. 5. to the sum of B and D (4).

PROP. XXIII. THEOR.

Fig. 23. *If there be four magnitudes proportional (A to B as C to D), and the first be greater than the third, the second is greater than the fourth; and if equal, equal; and if less, less.*

Part 1. Let A be greater than C; B is also greater than D.

For, *if it be possible*, let D be equal to B; and since A is greater than C, and B equal to D, there can be taken a submultiple of C, which is contained in D oftener
(1) Cor. Prop. 8. than an equi-submultiple of A is contained in B (1);
(2) Def. 5. contrary to the hypothesis (2).

In the same manner it can be demonstrated that B is not less than D.

Therefore B is greater than D.

Part 2. Let A be equal to C; B is also equal to D.

(3) Hypoth. Since A is to B as C to D (3), as often as any submultiple of A is contained in B, so often an equi-sub-
(4) Cor. Prop. 2. multiple of C is contained in D (2); and therefore since A is equal to C, B is equal to D (4).

Part 3. If A be less than C, B is less than D.

It can be demonstrated, as in the first part, that B is neither equal to nor greater than D; therefore B is less than D.

Cor. If the second be greater than the fourth, the first is greater than the third; and if equal, equal; and if less, less.

PROP. XXIV. THEOR.

If there be four magnitudes proportional (A to B as C to D), and the first be greater than the second, the third is greater than the fourth; and if equal, equal; and if less, less. Fig. 23.

Part 1. Let A be greater than B; C is also greater than D.

For since A is greater than B, there is a submultiple of A which is contained in it oftener than in B (1), and therefore an equi-submultiple of C is contained in it oftener than in D (2); and therefore C is greater than D (3).

(1) Prop. 1.
(2) Hypoth. & Def. 5.
(3) Prop. 5.

Part 2. Let A be equal to B; C is also equal to D.

For, since A is equal to B, a submultiple of A is contained in it as often as in B (1); but as often as a submultiple of A is contained in B, so often an equi-submultiple of C is contained in D (2); therefore as often as a submultiple of C is contained in C, so often it is contained in D; and therefore C is equal to D (4). (4) Prop. 2.

Part 3. Let A be less than B; C is also less than D.

For, since A is less than B, there is a submultiple of A, which is contained in B oftener than in A (5); therefore an equi-submultiple of C is contained in D oftener than in C (2); and therefore D is greater than C (3). (5) Prop. 4.

Cor. If the third be greater than the fourth, the first is greater than the second; and if equal, equal; and if less, less.

PROP. XXV. THEOR.

Fig. 24. *If there be four magnitudes proportional (A to B as C to D), and the first be less than the second, the first is to the excess of the second above it, as the third to the excess of the fourth above it.*

Take a and c equi-submultiples of A and C; and a is contained in B as often as c is contained in D (1); therefore a is contained in the excess of B above A, as often as c is contained in the excess of D above C; and therefore A is to the excess of B above A, as C to the excess of D above C (2).

(1) Hypoth. & Def. 5. (2) Def. 5.

Cor. 1. In the same manner it can be proved, that, if there be four magnitudes proportional, and the first be greater than the second, the first is to its excess above the second, as the third to its excess above the fourth.

Fig. 24. Cor. 2. If there be four magnitudes proportional, A to B as C to D, the difference between the first and second is to the second, as the difference between the third and fourth to the fourth.

For, since A is to B as C to D, B is to A as D to C (1), therefore B is to the difference between itself and A, as D to the difference between itself and C (2); and therefore the difference between A and B is to B, as the difference between C and D to D (1).

(1) Prop. 20. (2) Cor. pr. & Prop. 25.

PROP. XXVI. THEOR.

Fig. 24. *If there be four magnitudes proportional (A to B as C to D), any submultiple (a) of the first is to the second, as the equi-submultiple (c) of the third is to the fourth.*

Take x and z any equi-submultiples of a and c, and they must be equi-submultiples of A and C (1); therefore as often as x is contained in B, so often z is contained in D (2); and therefore a is to B as c to D (3).

(1) Prop. 7. (2) Hypoth. & Def. 5. (3) Def. 5.

PROP. XXVII. THEOR.

If there be four magnitudes proportional (A to B as C to D), the first is to any submultiple (b) of the second, as the third to the equi-submultiple (d) of the fourth. Fig. 25.

Since A is to B as C to D, B is to A as D to C (1); therefore, if there be taken *b* and *d* equi-submultiples of B and D, *b* is to A as *d* to C (2); and therefore A to *b* as C to *d* (1). (1) Prop. 20. (2) Prop. 26.

Cor. 1. If there be taken any equi-submultiples *a* and *b* of two magnitudes A and B, A is to *a* as B to *b*.

Cor 2. If there be four magnitudes proportional, A to B as C to D, and there be taken any equi-submultiples *a* and *c* of the antecedents, and also any equi-submultiples *b* and *d* of the consequents, *a* is to *b* as *c* to *d*. Fig. 26.

Since A is to B as C to D, and *a* and *c* are equi-submultiples of A and C, *a* is to B as *c* to D (1); and therefore, since *b* and *d* are equi-submultiples of B and D, *a* is to *b* as *c* to *d* (2). (1) Prop. 26. (2) Prop. 27.

PROP. XXVIII. THEOR.

Magnitudes (a and b) have the same ratio to one another which their equi-multiples have (AC and BD). Fig. 27.

Since AC and BD are equi-multiples of *a* and *b*, there are as many parts in AC equal to *a*, as there are in BD equal to *b*; divide AC into the parts Am, mn, nC, each equal to *a*, and BD into the parts Bo, os, sD, each equal to *b*.

Am is to Bo, as mn to os (1); and mn is to os as nC to sD (1); therefore as one antecedent Am is to one consequent Bo, so is the sum of all the antecedents, or AC, to the sum of all the consequents, or BD (2); but *a* is to *b* as Am to Bo (1); therefore *a* is to *b* as AC to BD (3). (1) Cor. Prop. 14. (2) Prop. 22. (3) Prop. 18.

Cor. 1. If there be four magnitudes proportional, *a* to *b* as *c* to *d*, and there be taken equi-multiples A and B of *a* and *b*, and also any equi-multiples C and

D of c and d, A is to B as C to D; as is evident from prop. 28. and prop. 18.

Cor. 2. Also if A is to B as C to D, and there be taken any equi-submultiples a and b of A and B, and any equi-submultiples c and d of C and D, a is to b as c to d.

PROP. XXIX. THEOR.

Fig. 26. *If there be four magnitudes proportional (a to b as c to d), any multiple (A) of the first is to the second, as the equi-multiple (C) of the third is to the fourth.*

(1) Cor. 1. Prop. 28.
(2) Prop. 27.

Let B be the same multiple of b that A is of a, and D the same multiple of d; A is to B as C to D (1); and therefore A to b as C to d (2).

PROP. XXX. THEOR.

Fig. 26. *If there be four magnitudes proportional (a to b as c to d), the first is to any multiple (B) of the second, as the third is to the equi-multiple (D) of the fourth.*

(1) Cor. 1. Prop. 28.
(2) Prop. 26.

Let A and C be the same multiples of a and c that B is of b; A is to B as C to D (1), and therefore a is to B as c to D (2).

PROP XXXI. THEOR.

Fig. 26. *If there be four magnitudes proportional (a to b as c to d), and there be taken any equi-multiples (A and C) of the antecedents, and any equi-multiples (B and D) of the consequents, these multiples are proportional.*

(1) Prop. 29.
(2) Prop. 30.

Because a is to b as c to d, and A and C are equi-multiples of a and c, A is to b as C to d (1); therefore, since B and D are equi-multiples of b and d, A is to B as C to D (2).

PROP. XXXII. THEOR.

If there be four magnitudes proportional (a to b as c to d) any equi-multiples whatsoever (A and C) of the first and third are either both equal to any assumed equi-multiples (B and D) of the second and fourth, or are both greater than them, or both less. Fig. 26. See N.

Because A and C are equi-multiples of the antecedents (1), and B and D equi-multiples of the consequents (1); A is to B as C to D (2); therefore, if A be greater than B, C is greater than D; and if equal, equal: and if less, less (3). (1) Hypoth. (2) Prop. 31. (3) Prop. 24.

PROP. XXXIII. THEOR.

If there be four magnitudes of the same kind proportional (A to B as C to DO), they are also proportional when taken alternately (A to C as B to DO). Fig. 28.

If there be taken any equi-submultiples a and b of A and B, a is contained in C as often as b is contained in DO.

For, *if it be possible*, let a be contained in C oftener than b is contained in DO, and repeat b as often as a is contained in C, so as to make DY; DY is greater than DO (1); and b is a submultiple of DY, let x be the equi-submultiple of DO, and z of C. (1) Prop. 5.

Because x and b are equi-submultiples of DO and DY (2), and DO is less than DY, x is less than b (3); and because z and x are equi-submultiples of C and DO (2), z is to x as C to DO (4),; but C is to DO as A to B (5), therefore z is to x as A to B (6); but a and b are equi-submultiples of A and B, therefore a is to b as A to B (4), and therefore z is to x as a to b (6); but x is less than b, therefore z is less than a (7); therefore there is a submultiple of z, which is contained in C oftener than an equi-submultiple of a is contained in the same C (8); but z and x are equi-submultiples of C and DO (2), and therefore as often as any submultiple of z is contained in C, so often an equi-submultiple of x is (2) Constr. (3) Ax. 3. (4) Prop. 28. (5) Hypoth. (6) Prop. 18. (7) Cor. Prop. 23. (8) Prop. 8.

(9) Prop. 7. contained in DO (9); therefore there is a submultiple of x, which is contained in DO oftener than an equi-submultiple of a is contained in C; but DO is made up (2) Constr. of x repeated as often as a is contained in C (2), therefore there is no submultiple of x, which is contained in (10) Cor. 2. Prop. 7. DO oftener than the equi-submultiple of a is contained in C (10); but there is given one: which is absurd.

Therefore no submultiple of A is contained in C oftener than the equi-submultiple of B is contained in DO; and in the same manner it can be demonstrated, that no submultiple of B is contained in DO oftener than the equi-submultiple of A is contained in B; (11) Def. 5. therefore A is to C as B to DO (11).

PROP. XXXIV. THEOR.

Fig. 29. *If there be any number of magnitudes, and as many others which taken two by two are in the same ratio, and if their proportion be* ordinate; *they are* ex æquali *in the same ratio.*

First, let there be three magnitudes A, B and CO, and as many others D, E and F; and let A be to B as D to E, and B to CO as E to F: A is to CO as D is to F.

If there be taken any equi-submultiples a and d of A and D, d is contained in F as often as a is contained in CO.

For, *if it be possible,* let d be contained in F oftener than a is contained in CO, repeat a as often as d is contained in F, so as to make up CY; CY is greater than (1) Prop. 5. CO (1); and a is a submultiple of CY, let x be the equi-submultiple of CO, and z of F.

Because x and a are equi-submultiples of CO and CY (2) Constr. (3) Ax. 3. (2), and CO is less than CY, x is less than a (3); and (4) Hypoth. since B is to CO as E to F (4), CO is to B as F to E; but x and z are equi-submultiples of CO and F (2), (5) Prop. 26. therefore x is to B as z to E (5); and therefore a submultiple of z is contained in E as often as the equi-submultiple of x is contained in B; but x is less than a, therefore there is a submultiple of x, which is contained in B oftener than the equi-submultiple of a is contained (6) Prop. 8. in the same B (6), and therefore oftener than the equi-

submultiple d is contained in E (7); therefore there is a submultiple of z, which is contained in E oftener than the equi-submultiple of d is contained in E, and therefore z is less than d (8); but F was made up of z repeated as often as d is contained in F (2); and therefore z is either greater than d, or equal to it (9), but it was proved that z was less than d: which is absurd. (7) Hypoth. & Def. 5. (8) Prop. 9. (2) Constr. (9) Cor. 3. Prop. 7.

Therefore no submultiple of D is contained in E oftener than the equi-submultiple of A is contained in CO; and in the same manner it can be demonstrated, that no submultiple of A is contained in CO oftener than the equi-submultiple of D is contained in F: therefore A is to C as D is to F (10). (10) Def. 5.

Let there be four magnitudes, A, B, C, G, and as many others, D, E, F, H; and let A be to B as D to E, B to C as E to F, and C to G as F to H; A is to G as D to H. Fig. 30.

Because there are three magnitudes, A, B, C, and as many others D, E, F, which if taken two by two are in the same ratio, A is to C as D to F (11): but C is to G as F to H (4); therefore A is to G as D is to H (11). (11) Part pr.

In the same manner it can be demonstrated, whatever be the number of magnitudes given.

PROP. XXXV. THEOR.

If there be three magnitudes (A, B, C), and as many others (D, E, F), which if taken two by two are proportional but perturbate *(A to B as E to F, and B to C as D to F); and if in the first series the first magnitude be greater than the third, the first in the second series is also greater than the third.* Fig. 31.

Because D is to E as B to C (1), E is to D as C to B (2); but A is greater than C (1), therefore A has a greater ratio to B than C has to the same B (3), and therefore a greater ratio than E has to D (4): but A is to B as E to F (1), therefore E has a greater ratio to F than it has to D (5); and therefore D is greater than F (6). (1) Hypoth. (2) Prop. 20. (3) Prop. 16. (4) Cor. Prop. 19. (5) Prop. 19. (6) Prop. 17.

PROP. XXXVI. THEOR.

Fig. 32. *If there be three magnitudes* (*A, B, C*), *and as many others* (*D, E, F*), *which if taken two by two are proportional but* perturbate (*A to B as E to F, and B to C as D to E*); *and if there be taken equi-submultiples* (*a, b, d*) *of the first and second magnitudes in the first series, and of the first in the second series* (*of A, B and D*), *and also any equi-submultiples* (*c, e and f*) *of the third in the first series, and of the second and third in the second series* (*of C, E and F*); *these equi-submultiples of the magnitudes in the first and second series are also proportional* perturbate (*a to b as e to f, and b to c as d to e*).

(1) Hypoth. Because A is to B as E to F (1), and *a* and *b* are taken equi-submultiples of A and B, and *e* and *f* equi-submultiples of E and F; *a* is to *b* as *e* to *f* (2).

(2) Cor. 2. Prop. 28. And since B is to C as D to E (1), and *b* and *d* are equi-submultiples of B and D, and *c* and *e* equi-submultiples of C and E (1); *b* is to *c* as *d* to *e* (2).

PROP. XXXVII. THEOR.

Fig. 32. *If there be three magnitudes* (*A, B, C*), *and as many others* (*D, E, F*), *which if taken two by two are in the same ratio but* perturbate (*A to B as E to F, and B to C as D to E*); *and if there be taken equi-submultiples* (*a and d*) *of the first magnitude in each series* (*A and D*), *and also any equi-submultiples* (*c and f*) *of the last in each series* (*C and F*); *and if the submultiple* (*a*) *of the first in the first series be greater than the submultiple* (*c*) *of the last in the same series; the submultiple* (*d*) *of the first in the second series is also greater than the submultiple* (*f*) *of the last in the same series.*

Take *b* the same submultiple of B that *a* is of A, and also *e* the same submultiple of E that *f* is of F, there are three magnitudes, *a, b, c*, and as many others, *d, e, f*, which if taken two by two are proportional, but *perturbate* (1), and therefore if *a* be greater than *c*, *d* is greater than *f* (2).

(1) Prop. 36.
(2) Prop. 35.

PROP. XXXVIII. THEOR.

If there be any number of magnitudes, and as many others, which taken two by two are in the same ratio but perturbate, *they are* ex æquali *in the same ratio.* Fig. 33, 34.

First, let there be three magnitudes, A, B, and CO, and as many others, D, E, F; if A be to B as E to F, and B to CO as D to E, A is to CO as D is to F.

For if there be taken any equi-submultiples a and d of A and D, d is contained in F as often as a is contained in CO.

For, *if it be possible*, let d be contained in F oftener than a is contained in CO; repeat a as often as d is contained in F, so as to make up CY; CY is greater than CO (1); and a is a submultiple of CY, let x be the equi-submultiple of CO, and also z of F. (1) Part 5.

Because x and a are equi-submultiples of CO and CY, and CO is less than CY, x is less than a (2); and since z is the same submultiple of F that x is of CO (3), and d is contained in F as often as x is contained in CO (3), d is contained in F as often as z, a submultiple of F, is contained in the same F; therefore d is either less than z or equal to it (4). Therefore there are taken equi-submultiples a and d of A and D, and x and z also equi-submultiples of CO and F, and a is greater than x, but d not greater than z, which is impossible (5). (2) Ax. 3. (3) Constr. (4) Cor. 3. Prop. 7. (5) Prop. 37.

Therefore no submultiple of D is contained in F oftener than the equi-submultiple of A is contained in CO; and in the same manner it can be demonstrated, that no submultiple of A is contained in CO oftener than the equi-submultiple of D is contained in F: therefore A is to CO as D is to F.

Let there be four magnitudes, A, B, C, G, and as many others, D, E, F, H, and let A be to B as F to H, B to C as E to F, and C to G as D to E; A is to G as D is to H. Fig. 34.

Because there are three magnitudes A, B, C, and as many others E, F, H, which if taken two by two are proportional *perturbate*, A is to C as E to H (6), but C (6) Part pr.

(7) Hypoth. is to G as D to E (7): and therefore A is to G as D is
(6) Part pr. to H (6).

In the same manner it can be demonstrated, whatever be the number of magnitudes given.

PROP. XXXIX. THEOR.

Fig. 1. Plate 7. *If there be three magnitudes proportional (A to B as B to C), and as many others also proportional (D to E as E to F); and if the first magnitude in the first series be to the last (A to C), as the first to the last in the second series (D to F); the first is to the second in the first series (A to B), as the first to the second in the second series (D to E).*

For, *if it be possible,* let one of them A have a less ratio to B, than D has to E, and since B is to C as A
(1) Hypoth. to B, and E to F as D to E (1), B has a less ratio to C
(2) Prop.19. than E has to F (2); therefore there is a submultiple of B, which is contained in C oftener than the equi-sub-
(3) Def. 7. multiple of E is contained in F (3); let those submultiples be b and e, and take equi-submultiples a and d of A and D; because a and b are equi-submultiples of A
(4) Prop.28. and B, a is to b as A to B (4), and also d to e as D to E (4); therefore a has a less ratio to b than d has to e (1), and therefore x can be taken a submultiple of a, which is contained in b oftener than the equi-submultiple z of d is contained in e (3); and since x is contained in b oftener than z is contained in d, and b is contained in C oftener than d in F, x is contained in C
(5) Cor. Prop. 1. oftener than z is contained in F (5), but x and z are equi-submultiples of a and d, and a and d are equi-submultiples of A and D, therefore x and z are equi-sub-
(6) Prop. 7. multiples of A and D (6); therefore z is contained in F
(7) Hypoth. & Def. 5. as often as x is contained in C (7); but it was proved before, that x was contained in C oftener than z in F: which is absurd.

Therefore A has not a less ratio to B than D has to E: and in the same manner it can be demonstrated, that D has not a less ratio to E than A has to B; and therefore A is to B as D to E.

THE

ELEMENTS OF EUCLID.

BOOK VI.

DEFINITIONS.

1. Similar rectilineal figures are those which have all their angles respectively equal, and the sides about the equal angles proportional.

2. A right line is said to be cut in extreme and mean ratio, when the whole line is to the greater segment, as the greater segment is to the less.

3. The altitude of any figure is a right line drawn from the vertex perpendicular to the base.

4. A parallelogram described upon a right line is said to be applied to that right line.

5. A parallelogram, described upon a part of a right line is said to be applied to that line *deficient by a parallelogram;* that is, by the parallelogram which is described upon the remaining part.

6. When a given right line is produced, the parallelogram described upon the whole line is said to be applied to the given line *exceeding by a parallelogram;* that is, by the parallelogram which is described upon the produced part.

PROP. I. THEOR.

Fig. 1. Plate 5. Book 6. See N. *Triangles (ABC, DBG) and parallelograms (BA and CF), which have the same altitude, are to each other as their bases.*

Part. 1. Divide the base of the triangle ABC into any number of equal parts AF, FK, and KC, take on the base DG, as often as it is possible, the segments DI, IE equal to AF, and draw BF, BK, BI, and BE.

Since the right lines AF, FK, KC, DI and IE, are
(1) Constr. equal (1), and the triangles constructed upon them have
(2) Prop. 38. B. 1. the same altitude, the triangles are equal (2); and therefore whatever submultiple AF is of AC, the same is the triangle ABF of the triangle ABC; and the triangle ABF is contained in the triangle DBG as often as AF is contained in DG; and in the same manner it can be shewn, that as often as any other submultiple of AC is contained in DG, so often the equi-submultiple of ABC is contained in DBG: and therefore the
(3) Def. 5. B. 5. triangle ABC is to the triangle DBG as the base AC to the base DG (3).

Fig. 2. Part. 2. The parallelograms BA and CF, which have the same altitude, are to each other as their bases BC and CD.

For draw BA and AD; and since the triangles BAC
(4) Part. pr. and CAD have the same altitude, they are to one ano-
(5) Prop. 34. B. 1. ther as their bases BC and CD (4): but the parallelo-
(6) Prop. 28. B. 5. grams are double of these triangles (5), and therefore are to each other as BC and CD (6).

Cor. 1. Triangles or parallelograms, which have equal altitudes, are to each other as their bases.

For, if the bases of the triangles were placed *in directum*, and perpendiculars let fall from the vertices upon the bases, these perpendiculars must be parallel (1):
(1) Prop. 28. B. 1. but they are also equal (2); therefore a right line, join-
(2) Hypoth. ing the vertices, would be parallel to the line in which
(3) Prop. 33. B. 1. the bases are (3): and therefore it can be demonstrated, in the same manner as in the proposition, that the triangles are to one another as their bases.

The parallelograms are also as their bases, because
(4) Prop. 34, B. 1. they are double of triangles (4) which have equal altitudes, and are therefore in that ratio.

Cor. 2. Triangles ABC, DEF, and parallelograms AB, DE, which have equal bases AC and DF, are to each other as their altitudes. Fig. 3.

If in the triangles one angle be right, it is evident.

If not, draw BK and EG parallel to AC and DF, and through the points C and F draw CK and FG perpendicular to AC and DF, and join AK, DG.

Because the triangles ABC, AKC, are upon the same base, and between the same parallels, ABC is equal to AKC (1), and likewise DGF is equal to DEF; but if the right lines CK and FG be considered as the bases of the triangles AKC and DGF, their altitudes AC and DF are equal (2), and therefore the triangles are as their bases CK and FG (3); therefore the triangles ABC and DEF, which are equal to them, are as CK and FG; but these lines are equal to their altitudes (4); and the parallelograms are double of the triangles, and therefore are also to each other as CK and FG.

(1) Prop. 37. B. 1.
(2) Hypoth.
(3) Cor. pr.
(4) Prop. 34. B. 1. & Def. 5.

PROP. II. THEOR.

If a right line (DE) be drawn parallel to any side (AC) of a triangle (ABC), it divides the other sides, or those sides produced, into proportional segments; and the homologous segments are at the same side of the parallel line (DE). Fig. 4.

And if a right line (DE) divide the sides of a triangle, or those sides produced, into proportional segments, so that the homologous segments be at the same side of it, it is parallel to the remaining side (AC).

Part 1. Let DE be parallel to AC, and AD is to DB as CE is to EB.

For draw AE and DC, and since the triangles EAD and ECD are upon the same base ED, and between the same parallels ED and CA, they are equal (1); therefore AED has the same ratio to DEB which CDE has to the same EDB (2); but AED is to DEB as AD to DB (3), and CDE is to EDB as CE to EB (3), therefore AD is to DB as CE is to EB (4).

(1) Prop. 37. B. 1.
(2) Prop. 14. B. 5.
(3) Prop. 1.
(4) Prop. 18. B. 5.

Part 2. Let AD be to DB as CE to EB, and the right line DE is parallel to AC.

Let the same construction remain, and AD is to
(3) Prop. 1. DB as the triangle AED to the triangle DEB (3), and as CE to EB, so is the triangle CDE to the triangle
(4) Prop. 18. B. 5. EDB (3); but AD is to DB as CE to EB, therefore
(5) Prop. 15. B. 5. AED is to DEB as CDE to the same EDB (4); therefore AED is equal to CDE (5): but they are upon the
(6) Prop. 39. B. 1. same base DE, and at the same side of it, and therefore DE is parallel to AC (6).

Fig. 5. Cor. If there be drawn several right lines IO and FL, parallel to the same side BC of the triangle BAC, all the segments of the other sides are proportional.

Draw FQ parallel to AC. In the triangle BFQ,
(1) Prop. 2. BI is to IF as QS to SF (1); but on account of the parallelograms QO and SL, CO is equal to QS, and OL
(2) Prop. 34. B. 1. to SF (2); and therefore CO is to OL as BI to IF: and in the same manner it can be demonstrated, whatever be the number of parallels.

PROP. III. THEOR.

Fig. 6. See N. *A right line (AD) bisecting the angle of a triangle (BAC) divides the opposite side into segments (BD, DC) proportional to the conterminous sides (BA, AC).*

And if a right line (AD) drawn from any angle of a triangle divide the opposite side (BC) into segments (BD, DC) proportional to the conterminous sides (BA, AC), it bisects the angle.

Part 1. Draw through C a right line CE parallel to AD, until it meet the side BA produced to E.

Because the lines AD and EC are parallel, the angle
(1) Prop. 29. B. 1. BAD is equal to the internal angle at the same side AEC (1); therefore the angle DAC is equal to AEC
(2) Hypoth. & Ax. 1. B. 1. (2): but DAC is equal to the alternate angle ACE (1)
(3) Prop. 6. B. 1. therefore ACE and AEC are equal, and therefore the opposite sides AE and AC are equal (3): but since AD
(4) Prop. 2. is parallel to EC, EA is to AB as CD is to DB (4);

therefore, since EA and AC are equal, AC is to AB as CD is to DB.

Part 2. Let the same construction remain, and BA is to AE as BD to DC (4); but BD is to DC as BA to AC (5); therefore BA is to AE as BA to AC (6), and therefore AE and AC are equal (7), and the angle ACE is equal to AEC (8): but since AD and EC are parallel, the angle DAC is equal to the alternate angle ACE (1), and the angle BAD equal to the internal angle at the same side AEC (1); therefore, since AEC and ACE are equal, BAD and DAC are also equal, and therefore the right line AD bisects the angle BAC.

(4) Prop. 2. B. 5.
(5) Hypoth.
(6) Prop. 18. B. 5.
(7) Prop. 15. B. 5.
(8) Prop. 5. B. 1.
(1) Prop. 29. B. 1.

Cor. If the right line bisecting the vertical angle of a triangle bisect the base, the triangle is isoceles.

PROP. IV. THEOR.

In equiangular triangles (BAC and CDE), the sides about the equal angles are proportional; and the sides which are opposite to the equal angles are homologous. Fig. 7.

For if the sides BC and CE, which are opposite to the equal angles BAC and CDE, were so placed that they should form one right line, that the triangles should be at the same side, and that the equal angles BCA and CED be not conterminous, since the angles ABC and BCA are less than two right angles (1), and CED is equal to BCA, ABC and CED are less than two right angles, and therefore the lines BA and ED must meet if produced (2): let them meet in F; because the angles BCA and CED are equal (3), CA is parallel to EF (4), and, because the angles ABC and DCE are equal, CD is parallel to BF (4); therefore AFDC is a parallelogram, and the side AC equal to FD, and also AF equal to CD (5).

(1) Prop. 17. B. 1.
(2) Ax. 12 B. 1.
(3) Hypoth.
(4) Prop. 28. B. 1.
(5) Prop. 34. B. 1.

In the triangle BFE the line AC is parallel to FE, therefore BA is to AF, or to CD equal to AF, as BC to CE (6); and by alternation, AB is to BC as CD to CE; and since CD is parallel to BF, BC is to CE as FD, or AC equal to FD, to DE (6); and by alternation BC is to CA as CE to ED; therefore, since AB

(6) Prop. 2.

is to BC as DC to CE, and BC to AC as CE to ED,
(7) Prop.34. B. 5. ex æquali (7), AB is to AC as DC to DE; therefore the sides about the equal angles are proportional, and those which are opposite to the equal angles are homologous.

Schol. It is evident that the triangles BAC and CDE are similar; and also that the sides, which are opposite to the equal angles, are proportional.

Fig. 8 Cor. 1. If in any triangle ABC, a parallel DE be drawn to any side AC, the triangle cut off, DBE, is similar to the whole.

Because the angle B is common to both, and the angles BDE, BED, are equal to the internal angles BAC,
(1) Prop.29. B. 1. (2) Schol. pr. BCA (1), the triangle DBE is equiangular to ABC, and therefore similar to it (2).

Cor. 2. If there be drawn a parallel DE to any side, AC, of the triangle ABC, the right line BO, drawn from the opposite angle, divides the parallel lines into proportional parts.

Since DI is parallel to AO, AO is to DI as OB to
(1) Schol. Prop. 4. (2) Prop.18. B. 5. IB (1); also OC is to IE as OB to IB; therefore AO is to DI as OC to IE (2), and, by alternation, AO is to OC as DI is to IE.

PROP. V. THEOR.

Fig. 9. *If two triangles (ABC, DEF) have their sides proportional (BA to AC as ED to DF, and AC to CB as DF to FE) they are equiangular; and the equal angles are subtended by the homologous sides.*

At the extremities of any side DE, of either triangle DEF, let the angles EDG and DEG be constructed equal to the angles A and B at the extremities of the
(1) Cor. 2. Prop. 32. B. 1. side AB which is homologous to ED, and in the triangle DEG the remaining angle G is equal to the angle C in the triangle ABC (1).

(2) Constr. (3) Prop. 4. (4) Hypoth. Because the triangles ABC and DEG are equiangular (2), BA is to AC as ED to DG (3); but BA is to AC as ED to DF (4), therefore ED is to DG as ED

to DF (5), and therefore DG and DF are equal (6); in the same manner it can be demonstrated that EG and EF are equal; therefore the triangle EDG is equilateral to EDF and therefore equiangular to it (7); but the triangle EDG is equiangular to BAC (2), and therefore BAC is equiangular to EDF, and it is evident that the homologous sides subtend the equal angles. (5) Prop. 18. B. 5. (6) Prop. 15. B. 5. (7) Schol. Prop. 8. B. 1. (2) Constr.

PROP. VI. THEOR.

If two triangles (ABC, DEF) have one angle in each equal (A equal to D), and the sides about the equal angles proportional (BA to AC as ED to DF); the triangles are equiangular, and the equal angles are subtended by the homologous sides. Fig. 9. See N.

At the extremities of either of the sides about the equal angles, DE, in the triangle DEF, let the angles EDG and GED be constructed equal to the angles A and B, at the extremities of the side AB which is homologous to ED; and in the triangle DEG the remaining angle G is equal to the remaining angle C in the triangle ABC (1). (1) Cor. 2. Prop. 32. B. 1.

Since the triangle ABC is equiangular to DEG (2), BA is to AC as ED to DG (3); but BA is to AC as ED to DF (4), therefore ED is to DG as ED to DF (5), and therefore DG and DF are equal (6): the angles EDG and EDF are also equal, because each of them is equal to the angle A (7), and the side ED is common to both, therefore the triangle EDF is equiangular to EDG (8); but BAC is equiangular to EDG (2), therefore BAC is equiangular to EDF; and it is evident that the homologous sides subtend the equal angles. (2) Constr. (3) Prop. 4. (4) Hypoth. (5) Prop. 18. B. 5. (6) Prop. 15. B. 5. (7) Hypoth. & Constr. (8) Prop. 4. B. 1.

PROP. VII. THEOR.

Fig. 10. *If two triangles (ABC, DEF) have two sides in the one proportional to two sides in the other (BA to AC as ED to DF); the angles opposite to the homologous sides (AC and DE) equal (B equal to E); and each of the remaining angles (C and F) either less or not less than a right angle; the triangles are equiangular, and those angles are equal about which the sides are proportional.*

Because the angles BAC and EDF are equal. For, *if it be possible,* let one of them BAC be greater than the other, and at the point A, with the right line AB, make the angle BAG equal to the less angle EDF.

In the triangles DEF, ABG, the angles E and B are equal (1), and EDF and BAG are also equal (2), therefore EFD is equal to BGA (3); therefore the triangles are equiangular, and BA is to AG as ED to DF (4), but BA is to AC as ED to DF (1); therefore BA is to AG as BA is to AC (5), and therefore AG is equal to AC (6); therefore the angle ACG is equal to AGC (7), and each of them acute (8); therefore if ACG and DFE be given not less than a right angle, ACG is acute and not acute at the same time; which is absurd; but if ACG and DFE be given less than a right angle, as the angle AGC is acute, the angle AGB is obtuse, therefore DFE which is equal to it must also be obtuse; but it is given less than a right angle; which is absurd.

(1) Hypoth.
(2) Constr.
(3) Cor. 2. Prop. 32. B. 1.
(4) Prop. 4.
(5) Prop. 18. B. 5.
(6) Prop. 15. B. 5.
(7) Prop. 5. B. 1.
(8) Cor. Prop. 17. B. 1.

The angle BAC therefore is not greater than EDF: and in the same manner it can be demonstrated, that EDF is not greater than BAC, they are therefore equal; and since the angles ABC and DEF are also equal (1), the triangles are equi-angular (3), and therefore have the sides about the equal angles proportional (4).

Schol. The triangles are similar, if it be given only that one of the angles C or F is a right angle.

PROP. VIII. THEOR.

In a right-angled triangle (ABC), if a perpendicular (BF) be let fall from the right angle upon the opposite side, it divides the triangle into parts which are similar to the whole and to one another. Fig. 11.

In the triangles AFB, ABC, the angle AFB is equal to the angle ABC (1), and the angle A is common to both, therefore the remaining angle ABF is equal to the remaining one C (2), and the triangles are equiangular; therefore the sides about the equal angles are proportional (3), and the triangles are similar (4).

(1) Hypoth.
(2) Cor. 2. Prop. 32. B. 1.
(3) Prop. 4
(4) Def. 1.

In the same manner it can be demonstrated, that the triangle BFC is similar to the triangle ABC.

Since the angle ABF is equal to the angle C, and the angles AFB and BFC are also equal, the remaining angles A and FBC are equal, and the triangles ABF and BCF are equiangular; therefore the sides about the equal angles are proportional (3), and therefore the triangles are similar.

Cor. Hence it is evident, that in every right-angled triangle the perpendicular BF is a mean proportional between the segments AF and FC of the side upon which it falls; and that the remaining sides AB and BC are mean proportionals between the conterminous segments AF and FC, and the whole side AC; and also, that the sides AC, AB, BC, and the perpendicular BF, are proportional. Fig. 11.

If the lines be given proportional, the triangle is right-angled; as is evident from prop. 6, and 7.

PROP. IX. PROB.

From a given right line (AB) to cut off any part required. Fig. 12.

From either extremity A, of the given line, draw AD making any angle with AB; in it take any point C, and make AD the same multiple of AC that AB is of

the required part; join BD, and draw through C a right line CI parallel to BD: AI is the part required.

(1) Cor. 1. Prop. 4. For AI is to AB as AC to AD (1), therefore what-
(2) Prop. 13. B. 5. ever submultiple AC is of AD, AI is the same submultiple of AB (2).

PROP. X. PROB.

Fig. 12. *To divide a given right line (AB) similarly to a given divided line (FG).*

From either extremity A, of the given line AB, draw AC, making any angle with it; take AD, DI and IL, equal to the parts of the divided line, FP, PR, and
(1) Prop. 3. B. 1. RG (1); join LB, and draw through I and D lines IK and DO parallel to LB.

Since in the triangle BAL the lines KI and OD are
(2) Cor. Prop. 2. parallel to BL, BK is to KO as LI to ID (2), or as GR to RP (3); and KO is to OA as ID to DA (2), or
(3) Constr. as RP to PF (3); and therefore the given line AB is divided similarly to FG.

PROP. XI. PROB.

Fig. 14. See N. *To find a third proportional to two given right lines (AB and FG).*

At either extremity of the given line AB draw AE making any angle with it, take AC equal to the other given line GF, and join BC; in AB produced take BD equal to FG, and through D draw DE parallel to BC, CE is the third proportional to AB and FG.

For, in the triangle DAE, BC is parallel to DE,
(1) Prop. 2. therefore AB is to BD as AC to CE (1); but BD and
(2) Constr. AC are equal to FG (2), therefore AB is to FG as FG is to CE.

PROP. XII. PROB.

Fig. 15. *To find a fourth proportional to three given lines (F, E and G).*

Draw two lines AD and AI making any angle; in AD take AB and BD equal to F and E, and in AI

take AC equal to G, join BC, and draw through D DI parallel to BC; CI is the fourth proportional to F, E and G.

For in the triangle DAI, BC is parallel to DI, therefore AB is to BD as AC to CI (1); but the given lines F, E and G, are equal to AB, BD and AC (2), therefore F is to E as G is to CI. (1) Prop. 2. (2) Constr.

PROP. XIII. PROB.

To find a mean proportional between two given right lines (E and F). Fig. 16. See N.

Draw any right line AC, take in it AB and BC equal to E and F; bisect AC in D; from the centre D with the radius DA describe a semicircle AIC, and through B draw BI perpendicular to AC, and meeting the circumference in I: BI is the mean proportional between E and F.

Draw AI and IC. Since in the triangle AIC the angle I is right (1), and IB is a perpendicular drawn from it to the opposite side; IB is a mean proportional between AB and BC (2), and therefore between the given lines E and F, which are equal to AB and BC (3). (1) Prop. 31. B. 3. (2) Cor. Prop. 8. (3) Constr.

Cor. 1. In the same manner mean proportionals can be found between the mean and the given lines, whence arises a series of five lines continually proportional; and then, if mean proportionals be found between the adjacent terms in this series, a series of nine proportionals arises; and so on. The number of proportionals in any series is one less than double the number in the preceding series.

Cor. 2. If the sum of the extremes of three proportionals, AC, be given, and the mean L, the extremes themselves can be found: for describe upon AC the semicircle AIC; draw CG perpendicular to AC, and equal to the mean L; draw GK parallel to AC; and from the point I, where it meets the circle, let fall a perpendicular IB upon AC: this will divide AC into parts AB and BC, the extremes sought for. Fig. 16.

Draw AI and IC. The angle AIC is right; therefore IB is a mean proportional between AB and BC

(1) Cor. Prop. 8. (1); but IB is equal to CG, and therefore equal to the given mean proportional L.

PROP. XIV. THEOR.

Fig. 17. *Equal parallelograms (AD and GC), which have one angle in each equal, have the sides about the equal angles reciprocally proportional (AB to BC as GB to BD).*

And parallelograms, which have one angle in each equal, and the sides about the equal angles reciprocally proportional, are equal.

Part 1. If the sides AB and BC were so placed that they should make one right line, and that the equal angles be vertically opposite; since ABD and DBC are equal to two right angles (1), and GBC is equal to ABD (2), GBC and DBC are equal to two right angles, and therefore GB and DB form one right line (3): complete the parallelogram DC.

(1) Prop. 13. B. 1. (2) Hypoth. (3) Prop. 14. B. 1.

Since the parallelograms AD and GC are equal (2), AD is to DC as GC to DC (4); but AD is to DC as AB to BC (5), and GC is to DC as GB to BD (5); therefore AB is to BC, as GB to BD (6).

(4) Prop. 14. B. 5. (5) Prop. 1. (6) Prop. 18. B. 5.

Part 2. Let the same construction remain: AD is to DC as AB to BC, and GC is to DC as GB to BD; but AB is to BC as GB to BD (7); therefore AD is to DC as GC to DC (6); and therefore the parallelogram AD is equal to the parallelogram GC (8).

(7) Hypoth. (8) Prop. 15. B. 5.

PROP. XV. THEOR.

Fig. 18. *Equal triangles, which have one angle in each equal (ABD equal to CBL), have the sides about the equal angles reciprocally proportional (AB to BC as LB to BD).*

And two triangles (ABD and CBL), which have one angle equal, and the sides about the equal angles reciprocally proportional, are equal.

Part. 1. If two of the sides AB and BC about the equal angles were so placed that they should form one

right line, and that the equal angles be vertically opposite, then since ABD and DBC are equal to two right angles (1), and LBC is equal to ABD (2), DBC and LBC are equal to two right angles; therefore DB and BL form one right line (3): join DC. (1) Prop. 13. B. 1. (2) Hypoth. (3) Prop. 14. B. 1.

Since the triangles ABD and LBC are equal (2), ABD is to DBC as LBC is to the same DBC (4); but ABD is to DBC as AB to BC (5); and LBC is to DBC as LB to DB (5); therefore AB is to BC as LB is to BD (6). (4) Prop. 14. B. 5. (5) Prop. 1. (6) Prop. 18. B. 5.

Part 2. Let the same construction remain; and ABD is to DBC as AB to BC, and LBC is to DBC as LB to DB (5); but AB is to BC as LB to BD (2); therefore ABD is to DBC as LBC to DBC (6); and therefore ABD is equal to LBC (7). (7) Prop. 15. B. 5.

Cor. 1. From this proposition and the preceding it is evident, that any triangles or parallelograms are equal, which have their bases and altitudes reciprocally proportional.

Cor. 2. If two triangles ABC and DEF have two sides reciprocally proportional to two, AB to EF as ED to AC, and the angles contained by those sides together equal to two right angles, the triangles are equal. Fig. 19.

Let the parallelograms AL and DF be completed: since the angles BAC and DEF together are equal to two right angles (1), and BAC and ACL are also equal to two right angles (2), DEF and ACL are equal; but BA is to EF as ED to AC (1), and LC is equal to AB (3), therefore CL is to EF as ED to AC; and the angles ACL and DEF are equal; therefore the parallelograms AL and DF are equal (4), and therefore the triangles ABC and DEF, which are halves of the parallelograms (3), are also equal. (1) Hypoth. (2) Prop. 29. B. 1. (3) Prop. 34. B. 1. (4) Prop. 14.

Cor. 3. In the same manner it can be proved, that if two equal triangles ABC and DEF have two angles BAC and DEF that are together equal to two right angles, the sides about them are reciprocally proportional.

PROP. XVI. THEOR.

Fig. 20. *If four right lines be proportional (A to B as C to D), the rectangle under the extremes (A and D) is equal to the rectangle under the means (B and C).*

And if the rectangle under the extremes be equal to the rectangle under the means, the right lines are proportional.

Part 1. Draw AE and GC equal to D and C; and erect AF and CK perpendicular to them, and equal to A and B; complete the rectangles EF and GK.

Because in the parallelograms EF and GK, the an-
(1) Hypoth. & Constr. gles A and C are equal, and the sides about them reci-
(2) Prop. 14. procally proportional (1), EF is equal to GK (2).

Part. 2. Let the same construction remain; because
(3) Hypoth. the parallelograms EF and GK are equal (3), and the angles A and C are equal, AF is to CK as GC to AE
(4) Constr. (2); and therefore A is to B as C to D (4).

Cor. 1. If four right lines be proportional, any parallelogram under the extremes is equal to an equiangular parallelogram under the means.

Fig. 21, 22. See N. Cor. 2. In any triangle, the square of the line bisecting any angle, internal or external, is equal to the difference between the rectangle under the sides about that angle, and the rectangle under the segments of the side which the bisecting line meets.

Let a circle be circumscribed about the given triangle BAC: let the bisecting line AD be produced till it meet the circumference in E; and draw EC.

(1) Hypoth. Since the angles BAD and EAC are equal (1), and
(2) Prop. 21, ABD and AEC are also equal (2), the triangles BAD
& 22. B. 3. and EAC are equiangular; therefore BA is to AD as
(3) Prop. 4. AE to AC (3), and the rectangle under BA and AC
(4) Prop. pr. is equal to the rectangle ander EA and AD (4); but the square of AD is the difference between the rectangle under EA and AD, and the rectangle under ED
(5) Prop. 3. B. 2. and DA (5), the rectangle under EA and AD is equal to the rectangle under BA and AC, and the rectangle
(6) Prop. 35, under ED and DA is equal to the rectangle under
& 36. B. 3. BD and DC (6); therefore the square of DA is equal

to the difference between the rectangle under the sides BA and AC, and the rectangle under the segments of the base BD and DC.

Cor. 3. If from any angle of a triangle a perpendicular be let fall on the opposite side, or the opposite side produced, the rectangle under the sides containing that angle is equal to the rectangle under the perpendicular and the diameter of the circle circumscribed about the triangle. Fig. 29.

Let a diameter AE be drawn, and join EC.

Since the angle ADB is right, it is equal to the angle ACE in a semicircle (1); and the angles ABC and AEC are also equal, because they stand upon the same arch AC (2); therefore the triangles BAD and EAC are equiangular, and BA is to AD as EA to AC (3); and therefore the rectangle under BA and AC is equal to the rectangle under AD and AE (4).

(1) Prop. 31. B. 3.
(2) Prop. 21. B. 3.
(3) Prop. 4.
(4) Prop. pr.

Cor. 4. In any quadrilateral figure ABCD inscribed in a circle, the rectangle under the diagonals AC and BD is equal to the sum of the rectangles under the opposite sides, under AB and CD, and under BC and AD. Fig. 25.

Make the angle ABE equal to CBD; and since the angles ABE and DBC are equal, and also BAE and BDC (1), the triangles ABE and DBC are equiangular; therefore AB is to AE as DB to DC (2); and the rectangle under AB and DC is equal to the rectangle under AE and DB (3): since the angle ABE is equal to CBD (4), if the common angle EBD be added to both, ABD and CBE are equal; also the angles BDA and BCE are equal (1), therefore the triangles ADB and ECB are equiangular, and AD is to DB as EC to BC (2), and therefore the rectangle under AD and BC is equal to the rectangle under DB and EC (3); but the rectangle under AB and DC is equal to the rectangle under DB and AE, and therefore the rectangles under AB and DC, and under AD and BC, are equal to the rectangle under DB and AC (5).

(1) Prop. 21. B. 3.
(2) Prop. 4.
(3) Prop. pr.
(4) Constr.
(5) Prop. 1. B. 2.

PROP. XVII. THEOR.

Fig. 24. *If three right lines be proportional (A to B as B to C), the rectangle under the extremes is equal to the square of the mean.*

And if the rectangle under the extremes be equal to the square of the mean, the three right lines are proportional.

Part 1. Assume a line D equal to B, and A is to B as D to C (1); therefore the rectangle under A and C is equal to the rectangle under B and D (2), and therefore equal to the square of B.

(1) Prop. 14. B. 5.
(2) Prop. 16.

Part 2. Assume a line D equal to B; the rectangle under A and C is equal to the rectangle under D and B; therefore A is to B as D to C (2), and therefore A is to B as B to C (1).

Cor. 1. If three right lines be proportional, any parallelogram under the extremes is equal to the equiangular parallelogram under the mean.

Fig. 16. Cor. 2. A given right line AC can be divided into segments AB and BC, the rectangle under which shall be equal to a given square, by dividing it so that the side of the square shall be a mean proportional between the segments of the given line (1): it is evident this cannot be done, when the side of the given square is greater than half the given line.

(1) Cor. 2. Prop. 13.

Cor. 3. The rectangle under any two lines is a mean proportional between their squares; for it is equal to the square of the mean proportional between the lines: but the squares of proportional lines are proportional, as will be shewn in prop. 22.

PROP. XVIII. PROB.

Fig. 25. *On a given right line (AB) to construct a rectilineal figure, similar to a given one (FGIKL) and similarly posited.*

Draw FI and FK; make at the extremities of the line AB the angles BAC and ABC equal to LFK and

FLK; let the lines AC and BC meet in C, and the angle BCA is equal to LKF (1); in the same manner construct upon AC a triangle equiangular with FKI, and so on. (1) Cor. 2. Prop. 32. B. 1.

The angles ABC and FLK are equal (2); BCD and LKI are also equal, because BCA is equal to LKF, and ACD to FKI (2): and in the same manner it can be proved that the angles in the figure AEDCB are severally equal to the angles in the figure FGIKL, therefore the figures AEDCB and FGIKL are equiangular; but since the triangles ABC and FLK are equiangular (2), AB is to BC as FL to LK (3), and also BC to CA as LK to KF (3): also ACD and FKI are equiangular; therefore CA is to CD as KF to KI (3), and therefore *ex æquali* (4) BC is to CD as LK to KI: and in the same manner it can be proved, that the sides about the other equal angles are proportional; and since the figures AEDCB and FGIKL are also equiangular, they are similar (5). (2) Constr. (3) Prop. 4. (4) Prop. 34. B. 5. (5) Def. 1.

PROP. XIX. THEOR.

Similar triangles (ABC, FIL) are to each other in the duplicate ratio of their homologous sides. Fig. 26.

Take a third proportional KC to the homologous sides AC and FL, and join BK.

Since AC is to CB as FL to LI (1), by alternation AC is to FL as CB to LI (2); but AC is to FL as FL to CK (3), therefore CB is to LI as FL to CK (4); and the angle C is equal to the angle L, therefore the triangle KBC is equal to FIL (5), and ABC has to both the same ratio (6); but ABC is to KBC as AC to KC (7), therefore ABC, is to FIL as AC to KC, or in the duplicate ratio of AC to FL (8). (1) Hypoth. (2) Prop. 33. B. 5. (3) Constr. (4) Prop. 18. B. 5. (5) Prop. 15. (6) Prop. 14. B. 5. (7) Prop. 1. (8) Def. 10. B. 5.

PROP. XX. THEOR.

Fig. 25. *Similar polygons may be divided into similar triangles, equal in number and proportional to the polygons: and the polygons are to each other in the duplicate ratio of their homologous sides.*

(1) Hypoth. & Def. 1.
(2) Prop. 6.

Part 1. For the angles G and E are equal, and the sides about them proportional (1), therefore the triangles FGI and AED are similar (2); since the angles GIF and EDA are equal, and also the angles GIK and EDC (1), the remainders FIK and ADC are equal; and since FI is to IG as AD to DE, and IG to IK as DE to DC (1), *ex æquali* FI is to IK as AD to DC (3); and therefore, as the angles contained by them are equal, the triangle FIK is similar to ADC (2): and in the same manner it can be proved that all the other triangles are similar.

(3) Prop. 34. B. 5.

(4) Prop. 19.
(5) Prop. 18. B. 5.
(6) Prop. 22. B. 5.

Part 2. As the triangle FGI is similar to AED, FGI is to AED in the duplicate ratio of FI to AD (4); also FIK is to ADC in the duplicate ratio of FI to AD (4); therefore FGI is to AED as FIK to ADC (5); and in the same manner it can be proved, that FIK is to ADC as FKL to ACB: therefore as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents (6), or the polygon FGIKL to the polygon AEDCB.

Part 3. Since the polygon FGIKL is to the polygon AEDCB as the triangle FGI to the triangle AED, and since FGI is to AED in the duplicate ratio of the side FG to AE (4), FGIKL is to AEDCB in the duplicate ratio of FG to AE (5).

See N. Cor. 1. If three right lines be proportional, any rectilineal figure on the first is to the similar rectilineal figure on the second, as the first to the third.

Cor. 2. Hence we can describe a rectilineal figure, which shall be to a given one in any assigned ratio; by finding a line, which shall be to the base of the given figure in the assigned ratio, and on a mean proportional between these lines describing a figure similar to the given one, and similarly posited.

PROP. XXI. THEOR.

Rectilineal figures (A and B), which are similar to the same figure (C), are similar also to each other. Fig. 27.

Since the rectilineal figures A and C are similar, they are equiangular, and have the sides about the equal angles proportional (1); and since the figures B and C are also similar, they are equiangular, and have the sides about the equal angles proportional (1); therefore the rectilineal figures A and B are also equiangular (2), and have the sides about the equal angles proportional (3), and are therefore similar.

(1) Def. 1.
(2) Ax. 1. B. 1.
(3) Prop. 18. B. 5.

PROP. XXII. THEOR.

If four right lines be proportional (AB to CD as EF to GH), the similar rectilineal figures similarly described on them are also proportional. Fig. 28. See N.

And if four similar rectilineal figures similarly described on four right lines be proportional, the right lines are also proportional.

Part 1. Take a third proportional X to AB and CD, and a third proportional O to EF and GH; since AB is to CD as EF to GH (1), CD is to X as GH to O (2); therefore *ex æquali* AB is to X as EF to O (3); but AKB is to CLD as AB to X (4), and EM to GN as EF to O (4), therefore AKB is to CLD as EM to GN.

Part 2. Let the same construction remain: AKB is to CLD as EM to GN (1); therefore AB is to X as EF to O (2), and therefore AB is to CD as EF to GH (5).

(1) Hypoth.
(2) Prop. 18. B. 5.
(3) Prop. 34. B. 5.
(4) Cor. 1. Prop. 20.
(5) Prop. 39. B. 5.

PROP. XXIII. THEOR.

Equiangular parallelograms (AD and GC) are to each other in a ratio compounded of the ratios of their sides. Fig. 17. See N.

If two of the sides AB and BC about the equal angles were so placed that they should form one right line,

since the angles ABD and DBC are equal to two right
(1) Prop. 13. B. 1. angles (1), and GBC is equal to ABD (2), GBC and
(2) Hypoth. DBC are equal to two right angles, and therefore GB
(3) Prop. 14. B. 1. and BD form one right line (3): complete the parallelogram BF.

Since the parallelogram AD is to BF as AB to BC
(4) Prop. 1. (4), and BF to BE as BD to BG (4), AD has to BE a
(5) Def. 12. B. 5. ratio compounded of the ratios of AB to BC, and of BD to BG (5).

Fig. 30. Cor. 1. Two right lines can be found, that shall be to each other in the ratio of the given parallelograms AD and BE. Take two lines I and K equal to BA and BC; to DB, BG and K, find a fourth proportional L; AD is to BF as I to K, and BF to BE as K to L, therefore *ex æquali* AD is to BE as I to L.

Cor. 2. Triangles, which have one angle equal, are to each other in a ratio compounded of the ratios of the sides containing the equal angles.

Cor. 3. Any parallelograms or triangles are to each other in a ratio compounded of the ratios of their bases and altitudes; for they are equal to rectangles, or to right-angled triangles, on the same bases and the same altitudes.

PROP. XXIV. THEOR.

Fig. 31. *In any parallelogram (AC) the parallelograms (AF and FC), which are about the diagonal, are similar to the whole and to each other.*

As the parallelograms AC and AF have a common
angle, they are equiangular (1): but, on account of the
(1) Cor. 2. Prop. 34. B. 1. parallels EF and BC, the triangles AEF and ABC are
similar (2), therefore AE is to EF as AB to BC (3);
(2) Cor. 1. Prop. 4. and the remaining sides are equal to AE, EF, AB and
(3) Def. 1. BC (4); therefore the parallelograms AF and AC have
(4) Prop. 34. B. 1. the sides about the equal angles proportional, and are therefore similar.

In the same manner it can be demonstrated, that the parallelograms AC and FC are similar.

Since therefore each of the parallelograms AF and
(5) Prop. 21. FC is similar to AC, they are similar to each other (5).

PROP. XXV. PROB.

To construct a rectilineal figure equal to a given one (A), and similar to another (B). Fig. 32.

On any side EF of the given figure B construct a rectangle EL equal to B (1), and on the side FL construct a rectangle FD equal to A (1): between the other sides EF and FG of these rectangles find a mean proportional CK (2); the figure described upon it similar to the given B, and similarly posited, is equal to the other given figure A.

(1) Cor. 1. Prop. 45. B. 1.
(2) Prop. 13.

For the rectangle EL is to the rectangle FD as EF to FG (3), or in the duplicate ratio of EF to CK (4); and therefore as the rectilineal figure B to the similar one upon CK (5); but EL is equal to B (4); therefore the rectilineal figure upon CK similar to B, and similarly posited, is equal to FD, and therefore equal to the given figure A (6).

(3) Prop. 1.
(4) Constr.
(5) Prop. 20.
(6) Prop. 23, B. 5.

Cor. Hence a rectilineal figure can be constructed similar to a given one, and equal to the sum or difference of two others, by first constructing a parallelogram equal to that sum or difference (1).

(1) Cor. 2 & 3 Prop. 45. B. 1.

PROP. XXVI. THEOR.

If similar and similarly posited parallelograms (AC and AF) have a common angle, they are about the same diagonal. Fig. 33.

For, *if it be possible,* let AIF be the diagonal of the parallelogram AF; and draw through I the right line IL parallel to AE.

Since the parallelograms AI and AF are about the same diagonal AIF, and have a common angle A, AI and AF are similar (1); therefore BA is to AL as EA to AG: but BA is to AD as EA to AG (2); therefore BA is to AL as BA to AD (3), and therefore AL is equal to AD (4): which is absurd. Therefore AIF is not the diagonal of AF; and in the same manner it can be demonstrated that no other line is, except ACF.

(1) Prop. 24.
(2) Hypoth.
(3) Prop. 18. B. 3.
(4) Prop. 15. B. 5.

PROP. XXVII. THEOR.

Fig. 34, 35. See N.

If any right line (AB) be bisected (in C), and cut unequally (in D), the parallelogram (FC), which is applied to the half, deficient by a figure (GB) similar to itself, is greater than the parallelogram (ED) applied to either of the other parts, deficient by a figure (KB) similar to the former (GB).

Fig. 34. First, let AD be the greater segment of AB; complete the parallelogram KI; and draw GB.

(1) Hypoth. (2) Prop. 26. (3) Prop. 43. B. 1. (4) Hypoth. & Prop. 36. B. 1.

Since GB and KB are similar (1), GB is the diagonal of both (2), therefore CK is equal to KI (3), and if DL be added to both, CL is equal to DI; but CL and CE are equal (4), therefore CE and DI are equal; add to both CK, DE is equal to the gnomon CLS, and therefore less than the parallelogram CI, therefore less than FC which is equal to CI.

Fig. 35. Now let AD be the less segment; complete the parallelogram GI; and draw KB.

(5) Hypoth. & Prop. 34. B. 1. (6) Prop. 36. B. 1.

Since the parallelograms KB and GB are similar (1), they are about the same diagonal (2), therefore the parallelograms DG and GI are equal (3); but the right lines FG and GL are equal (5), and therefore the parallelograms EG and GI are equal (6); but EG is greater than FK, therefore GI is greater than FK, and DG which is equal to GI is also greater than FK; add to both FD, and FC is greater than ED.

Schol. It is evident that the parallelogram KG is the excess of the parallelogram FC above ED, and that it is diminished when DC is diminished.

PROP. XXVIII. PROB.

Fig. 36.

To a given right line (AB) to apply a parallelogram equal to a given rectilineal figure (Z), and deficient by a figure similar to a given parallelogram (X). But the rectilineal figure must not be greater than the parallelogram applied to half the given line, whose defect is similar to the given parallelogram (X).

Bisect AB in E; describe upon AE a parallelogram AG similar to the given X; and complete APFB.

AG is either equal to or greater than the given rectilineal figure Z (1). If it be equal the problem is done. (1) Hypoth.
If it be greater, construct a parallelogram KLMN equal to its excess above Z, and similar to X (2); since (2) Cor. this parallelogram is less than AG, it is less than EF Prop. 25. which is equal to AG (3); but it is similar to it, and (3) Constr. therefore its sides KL and LM are less than the ho- & Prop. 36. mologous sides EG and GF of the parallelogram EF; B. 1. take away from these GK and GO, equal to KL and LM, and complete the parallelogram KGOI; this is similar to EF, since both are similar to X (4), and it is (4) Constr. also similarly posited, therefore KGOI and EF are about the same diagonal (5); draw their diagonal GIB, (5) Prop. 26. produce OI to S, and KI to M and N; since the parallelogram EF is equal to the sum of KLMN and Z (4), but KO is equal to KLMN, the gnomon ENO is equal to Z; but EI and IF are equal (6); therefore, if (6) Prop. 43. SN be added to both, EN and SF are equal; but since B. 1. AE and EB are equal, EN is equal to ME (7); there- (7) Prop. 36. fore ME and SF are equal, and therefore if EI be B. 1. added to both, MS is equal to the gnomon ENO; but ENO is equal to the given rectilineal figure Z, therefore MS is equal to the given Z, and its defect SN, since it is similar to the parallelogram EF (8), is also similar to (8) Prop. 24. X (9); and therefore the required problem is done. (9) Constr. & Prop. 21.
Schol. In this manner the parallelogram is con- Fig. 37. structed upon the greater segment of the given line AB; but it can also be constructed upon the less, if the lines GO and GK be taken away from the right line EG produced, and from GM; and having described the figure as in the proposition, AI is the required parallelogram, as can be easily demonstrated; for KO is the excess of AG above AI (1), and it is also its excess (1) Schol. above the given figure Z; therefore AI and Z are equal. Prop. 27.

PROP. XXIX. PROB.

To a given right line (AB) to apply a parallelogram equal to a given rectilineal figure (Z), and exceeding by a figure (BX) similar to a given parallelogram (X). Fig. 38.

Bisect AB in E; upon EB construct a parallelogram similar to the given parallelogram X, and construct a

parallelogram GH similar to the parallelogram EL,
(1) Cor. Prop. 22. and equal to the sum of EL and Z (1): since GH is greater than EL, its sides GK and KH are greater than the homologous sides of EL, which are FE and FL; on these sides produced take FN and FM equal to GK and KH, and complete the parallelogram NM: this is similar to GH, therefore similar to EL, and it is similarly posited, and therefore they are about the same diagonal; draw the diagonal FBX, through A draw AC parallel to EN, until it meet PN produced; since NM and GH are equal, and GH is equal to the
(2) Constr. sum of Z and EL (2), NM is also equal to the sum of Z and EL; take away from both EL, and the gnomon NOL is equal to Z; but since AE and EB are equal,
(3) Prop. 36. B. 1. the parallelograms AN and EP are equal (3), and also
(4) Prop. 43. B. 1. EP and BM are equal (4), therefore AN is equal to BM; add ON to both, and AX is equal to the gnomon NOL, and therefore equal to the given rectilineal figure Z; and its excess PO is similar to the parallelogram EL, and therefore similar to the given figure X.

PROP XXX. PROB.

Fig. 39. *To cut a given finite right line (AB) in extreme and mean ratio.*

(1) Prop. 46. B. 1. Describe BC the square of AB (1); to AC apply a parallelogram equal to BC, and exceeding by a figure AD
(2) Prop. 29. similar to BC (2): since AD is similar to BC, it is a square; since BC and CD are equal, if CE be taken away from both, BF and AD are equal; and they are
(3) Prop. 14. equiangular; therefore EF is to ED as EA to EB (3): but EF and ED are equal to AB and AE, therefore AB is to AE as AE to EB.

Otherwise thus.

Divide AB in E, so that the rectangle under AB and
(1) Prop. 11. B. 2. EB shall be equal to the square of AE (1); and AB is
(2) Prop. 17. to AE as AE to EB (2), therefore AB is cut in extreme and mean ratio.

PROP. XXXI. THEOR.

If any similar rectilineal figures be similarly described on the sides of a right-angled triangle (BAC), the figure, described on the side (BC) subtending the right angle, is equal to the sum of the figures upon the other sides. Fig. 40.

From the right angle draw a perpendicular AD to the opposite side: BC is to CA as CA to CD (1), therefore the figure upon BC is to the similar figure upon CA as BC to CD (2); and therefore the figure upon BC is to the difference between itself and the figure upon CA as BC to BD (3): but the figure upon BC is to the similar figure upon BA as BC to BD (4), therefore the figure upon BA is equal to the difference between the similar figures upon BC and CA (5); therefore the figure described upon the side subtending the right angle is equal to the sum of the similar figures upon the sides.

(1) Cor. Prop. 8. (2) Cor. 1. Prop. 20. (3) Cor. 1. Prop. 25. B. 5. (4) Prop. 20. & Cor. 5. Prop. 8. (5) Prop. 15. B. 5.

Cor. Given any number of similar rectilineal figures, a similar figures can be found equal to their sum, by Cor. 2. prop. 47. B. 1. and by Cor. 3. prop. 47. B. 1. a figure can be found, equal to the difference between two given similar figures.

PROP. XXXII. THEOR.

If two triangles (ABC, CDE) have two sides proportional (AB to BC as CD to DE), and be so placed at an angle that the homologous sides are parallel, and that the sides not homologous (CB and CD) form the angle at which they are placed, the remaining sides (AC and CE) form one right line. Fig. 41. See N.

Because AB and CD are parallel, the alternate angles B and BCD are equal (1); and also, since CB and ED are parallel, the angles D and BCD are equal (1), therefore B and D are equal; and since the sides about these angles are proportional (2), the triangles ABC

(1) Prop. 29. B. 1. (2) Hypoth.

(3) Prop. 6. and CDE are equiangular (3), therefore the angles ACB and CED are equal; but BCD is equal to CDE, and if DCE be added, ACD and DCE are together
(4) Prop. 32. B. 1. equal to CED, EDC and DCE; therefore ACD and
(5) Prop. 14. B. 1. DCE are equal to two right angles (4); and therefore AC and CE form one right line (5).

PROP. XXXIII. THEOR.

Fig. 42. See N. *In equal circles angles, whether they be at the centres (AFE and HOL) or at the circumferences (AGE and HNL), have the same ratio, that the arches on which they stand have to one another; so also have the sectors (AFE and HOL).*

First, let the given angles AFE and HOL be at the centres: suppose the arch ACE divided into any number of equal parts AC and CE, and take HI, IK and KL, equal to AC; and draw FC, OI and OK.

Since the arches AC, CE, HI, IK and KL are
(1) Constr. equal (1), the angles AFC, CFE, HOI, IOK and
(2) Prop. 27. B. 3. KOL are equal (2); therefore whatever submultiple the arch AC is of ACE, the same is the angle AFC of AFE; and as often as the arch AC is contained in HIKL, so often the angle AFC is contained in HOL;
(3) Def. 5. B. 5. and therefore ACE is to HIKL, as AFE is to HOL (3).

In the same manner it can be demonstrated, if the given angles AGE and HNL be at the circumferences, that AGE is to HNL as the arch ACE to the arch HIKL.

The sector AFE is to the sector HOL as the arch ACE to the arch HIKL.

Let the same construction remain, and since the arches AC, CE, HI, IK and KL, are equal (1), the sectors AFC, CFE, HOI, IOK and KOL, are equal
(4) Schol. Prop. 29. B. 3. (4); therefore whatever submultiple the arch AC is of ACE, the same is the sector AFC of AFE, and as often as the arch AC is contained in HIKL, so often the sector AFC is contained in HOL; and therefore ACE is to HIKL, as the sector AFE is to the sector HOL (3).

Cor. 1. Any angle AFE at the centre is to four right angles, as the arch upon which it stands to the whole circumference.

For the angle AFE is to a right angle as the arch on which it stands to the fourth part of the circumference (1), and therefore is to four right angles, as the arch to the whole circumference (2). (1) Prop. 33. (2) Prop. 36 B. 5.

Cor. 2. The arches of unequal circles, which subtend equal angles, are similar.

It is evident, that they have the same ratio to the whole circumference (1). (1) Cor. pr.

Cor. 3. It is evident, that the arches of similar segments are similar.

NOTES.

DEFINITION. B. 1.

WHAT is meant by a *point*, a *line*, and a *surface*, may easily be understood, by considering the nature of a solid or body. The boundaries of any solid are not parts of it, and therefore have no thickness, their only dimensions are length and breadth, and therefore they are *surfaces*. The boundaries of a surface have only length; if they had breadth they must be parts, not boundaries; they are therefore *lines*. The boundaries of lines want even length, they have therefore no dimensions, and are *points*.

DEF. 7. B. 1.

The learned Robert Simson gives a different definition of a plane surface. *A plane superficies is that in which, any two points being taken, the straight line between them lies wholly in that superficies:* this indeed is a well known property of a plane; but as it can easily be deduced from Euclid's definition compared with that of a right line, I thought it better not to make any change.

The young Student should be aware that this definition, and also the fourth, are not *definitions properly so called*, for a *right line* and a *plane surface* are simple ideas, which only admit *description*. It might be objected indeed to this definition of a *plane surface*, that there are planes, a circle, an ellipse, &c. which are not terminated by right lines: but if it be once understood what a plane is which is terminated by *right lines*, there can be no difficulty in conceiving that any part of it is also a plane, though that part be bounded by a *curve*.

DEF. 8. B. 1.

In the Greek copies this is the ninth definition, as it is preceded by a sort of general definition of an angle, *the inclination of two lines to one another in a plane, which meet together, but are not in the same direction:* This definition is omitted, as its only use appears to be in the doctrine of the angle of a semicircle and segment (prop. 16, and 31. B. 3.) which ought entirely to be rejected, as producing nothing but paradoxes and disputes, wholly unworthy of geometers.

DEF. 9. B. 1.

This definition, and the tenth, are not Euclid's, but are added, because the terms often occur, and require explanation.

DEF. 13. B. 1.

The definition which followed this, namely, *a term is the extremity of any thing*, is omitted as useless; the definitions of *a segment of a circle*, and of *an oblong*, are also omitted; because the former is one of the definitions of B. 3. and the latter is one of those of B. 2. where it is called a rectangle. Also the definitions of a *rhombus*, which has its four sides equal, but not its angles, and of a *rhomboid*, which has its opposite sides equal, but not the adjacent, are omitted; as no mention is made of them in the elements. Instead of them is substituted the definition of a parallelogram.

DEF. 17. B. 1.

R. Simson has rightly omitted the words which are usually added to this definition, *which also divides the circle into two equal parts;* as not belonging to the definition, being only a corollary from it. It can be demonstrated, by conceiving one of the parts into which the circle is divided to be applied to the other; for it is evident they must coincide, on account of the equality

of the radii. The same thing is easily deduced from prop. 31. and 24. B. 3. from the former of which it follows, that semicircles are similar segments; and from the other, that they are equal to one another.

POST. 3.

It should be remarked that the radius, with which the circle is described, must be the interval between the centre and some other point, not between any two points different from the centre; as is evident from the constructions of prop. 2, and 3. B. 1. which would be unnecessary, if the description of a circle from any point with a radius equal to any given line were conceded.

AX. 11. B. 1.

This axiom does not treat of angles adjacent to the same perpendicular, which are equal by def. 11; but of those which are formed by different perpendiculars with different right lines. By help of it we can demonstrate that two right lines have not a common segment. For *if it be possible*, let the lines BC and BD have a common segment AB, and let BE be perpendicular to the line ABC, if it be also perpendicular to ABD, the angles CBE and DBE are equal (1); which is absurd: but if not, let BF be perpendicular to the line ABD, and the angles ABF and ABE are equal (1); which is absurd. Plate 7. Fig. 3. (1) Ax. 11.

R. Simson demonstrates this in a corollary to prop. 11. B. 1; but his demonstration does not appear to me perfect: Through B he draws BE perpendicular to AB, and assumes that there can be but one perpendicular at that point; but this cannot be conceded, since, to erect a perpendicular, the line AB must first be produced, and if this can be done in different ways, there can be several perpendiculars at the point B; as is evident from the construction of prop. 11. B. 1: and therefore the whole demonstration fails.

AX. 12. B. 1.

Although it is a generally received principle, that right lines which are inclined to each other must meet if produced, yet since there are lines, which though constantly approaching to each other never meet, as a curve and its asymptote, this principle is too defective to be ranked among geometrical axioms. Many have endeavoured, but unsuccessfully, to demonstrate the doctrine of parallels without this axiom: each has assumed a principle not more evident than that which he had rejected. For those who are dissatisfied with Euclid's method, a new demonstration of the doctrine is given in N. prop. 27. B. 1.

By some editors the last three axioms are placed among the postulates. Although it must be confessed that they differ considerably from the others, and, depending as they do upon the definitions, can scarcely merit the name of *common notions;* yet, following the authority of the best editors, I have placed them among the axioms; and if (as most geometers assert) postulates refer to problems, axioms to theorems, they are rightly placed there.

PROP. 1. B. I.

Euclid assumes in the construction of this problem, that the circles intersect; it is impossible but part of the circle BCD, whose centre A is in the periphery of the circle ACE, must be outside that circle, and part within it.

PROP. 2. B. I.

According to the different situations of the point A, this construction somewhat changes: if the given point be in the given line, it is unnecessary to draw the line upon which the equilateral triangle is to be described: and if it be either extremity of the given line, the triangle also is unnecessary; and no part of the construction is required except the description of the circle

ACF, any line drawn to the periphery of which is equal to the given line : the point may be also so given, as to be the vertex of an equilateral triangle, of which the given line is the base, in which case it is evident that the line drawn to the extremity of the given line solves the problem; also another line equal to the given one can be drawn at the opposite side, in which case the radius of the circle, whose centre is in the vertex of the triangle, is only the produced part of the side : these different cases are designedly omitted by Euclid, who wrote for geometers, not for mechanics ; and was only anxious to make his constructions intelligible, wholly regardless of practice.

PROP. 3. B. 1.

In this proposition, as in the preceding, the position of the point A changes the construction: if it be at the extremity of the given line, the description of the circle, whose radius is the given line, is sufficient. It is possible that the line drawn from the point A may coincide with AB, in which case the description of the circle is unnecessary.

PROP. 4. B. 1.

Euclid has expressed this proposition in an absolute not an hypothetic form ; he says *the triangle ABC being applied to EDF, &c.* But as learners are apt to understand by these words some mechanical application, I thought it better to use the hypothetic form.

PROP. 8. B. 1.

In this proposition the equality only of the angles at the vertices B and F is demonstrated; for from it by prop. 4. can be deduced the equality of the other parts: it seemed useful however, to annex the scholium, that the same might also be shewn by this proposition.

PROP. 12. B. 1.

The given line must not be finite; for if it were, the problem might be impossible, as is evident from the figure, if C be the given point and the line finite AE.

PROP. 13. B. 1.

Euclid says *makes angles with it;* because a line drawn to the extremity of another stands upon it, but makes only one angle with it.

PROP. 20. B. 1.

Some writers contend that this proposition should be placed among the axioms; and that such was the opinion of Archimedes, who assumes that a right line is the shortest line between two points: but there he is comparing right lines with curves; Archimedes would never have assumed what Euclid had demonstrated: nor is every proposition, whose truth is very evident, to be placed among the axioms; as our aim should be to have the principles as few as possible.

This demonstration is not Euclid's, but some ancient geometer's.

PROP. 22. B. 1.

In this proposition, as well as in the first, Euclid assumes that the circles intersect; but this is easily demonstrated from the conditions of the problem, that any two of the given lines must be greater than the third: for, since the sum of the radii DG and EF is greater than DE, part of each circle must be within the other; and since the sum of DE and either of the others is greater than the third, part of each circle must be outside the other.

PROP. 24. B. 1.

In the construction of this proposition the Greek editors have omitted the words *which is not the greater,*

but these are absolutely necessary to prevent a diversity of cases, the point G falling sometimes above the line BC, sometimes below it, and sometimes upon it.

If the angle BAG be constructed with the greater side, and the point G fall either upon or above BC, the proposition can be demonstrated in the following manner:

1. Let the point G fall upon BC, and it is evident that BC is greater than BG. Fig. 6.

2. Let G fall above BC; since the sum of the lines AG and GB is less than the sum of AC and CB, but AG is equal to AC, BG must be less than BC. Fig. 7.

This latter case can be demonstrated otherwise thus: draw GC, and produce AG and AC; since the angles EGC and FCG are equal (1), and BGC greater than one of them, and BCG less than the other, BGC is greater than BCG, and therefore the side BC is greater than BG. Fig. 8.

(1) Prop. 5. B. 1.

PROP. 26. B. 1.

The line, equal to the side DE, ought to be cut off conterminous to the side BC.

PROP. 27. B. 1.

If the definition of parallel lines be changed, and an axiom laid down, which Euclid made use of, this proposition, the two following, and the 12th axiom, can be demonstrated in the following manner.

Definition.

Parallel right lines are those which however produced are always equidistant.

Axiom.

Any part of a finite right line can be taken so often as to exceed the whole.

Lemma 1.

Fig. 9. If two right lines AB and CD be parallel, any right line MN, which is perpendicular to one of them AB, is also perpendicular to the other CD.

Take any point A, and make NB equal to NA, draw AC and BD perpendicular to AB, and draw AM and BM.

In the triangles ANM and BNM the angles at N are equal, the sides NA and NB are also equal, and NM is common to both; therefore the angles NMA and NMB are equal, the sides MA and MB, and the angles NAM and NBM (1): take away the equal angles NAM and NBM from the equals NAC and NBD, the remainders MAC and MBD are equal, the sides MA and AC are equal to MB and BD; therefore the angles AMC and BMD are equal (1): but NMA and NMB are also equal; therefore the angles NMC and NMD are equal, and therefore NM is perpendicular to CD (2).

(1) Prop. 4. B. 1.
(2) Def. 11. B. 1.

Lemma 2.

Fig. 10. If two right lines AB and CD be perpendicular to the same line NM, they are parallel.

For, *if it be possible*, let CD not be parallel to AB, and through M draw PR parallel to AB; this line is perpendicular to NM (1), therefore the angle RMN is right, but DMN is also a right angle: which is absurd.

(1) Lem. 1.

Lemma 3.

Fig. 11. If two right lines AB and CD be parallel, and AC and BD be drawn perpendicular to them, the intercepted parts AB and CD are equal.

For since the right lines AC and BD are perpendicular to the same line AB, they are parallel (1), and therefore the lines AB and CD are equal (2).

(1) Lem. 2.
(2) Def. Parall.

PROP. 29. B. 1.

The right line EF, cutting the parallel right lines AB and CD, makes the alternate angles, AGH and GHD, CHG and HGB, equal; and the external angle equal to the internal and opposite at the same side, EGA to GHC and EGB to GHD; and the internal angles at the same side, AGH and CHG, BGH and DHG, equal to two right angles. Fig. 12.

If the line EF be perpendicular to the parallels, the proposition is evident.

But if not, draw HK and GL perpendicular to CD.

In the triangles LHG and KGH, the sides LG and HK (1), LH and GK (2) are equal, and GH common to both; therefore the angles LHG and KGH are equal, and therefore the angles CHG and BGH, which make up with them two right angles. (1) Def. parall. (2) Lem. 3.

Since the vertically opposite angles EGA and BGH are equal, and BGH is equal to the alternate angle CHG, the external angle EGA is equal to the internal GHC; and in the same manner it can be demonstrated that EGB and GHD are equal.

Since the angles CHG and DHG are equal to two right angles, and BGH is equal to the alternate angle CHG, DHG and BGH are equal to two right angles; and in the same manner it can be demonstrated, that GHC and AGH are equal to two right angles.

PROP. 27. B. 1.

If a right line MN, cutting two right lines CD and AB, make the alternate angles equal, CMN and MNB, those lines are parallel. Fig. 10.

For, *if it be possible*, let the line CD not be parallel to AB, and through M draw PR parallel to AB; the alternate angles PMN and MNB are equal: but CMN and MNB are equal, therefore PMN and CMN are equal: which is absurd.

PROP. 28. B. 1.

Fig. 14. If a right line EF intersect two right lines, AB and CD, and make an external angle equal to the internal and opposite upon the same side of the line, EGA to GHC, or EGB to GHD; or make the internal angles at the same side, AGH and CHG, or BGH and DHG, equal to two right angles, the two right lines are parallel to one another.

1. Let EGA and GHC be equal; and since EGA and BGH are equal, the alternate angles GHC and BGH are equal, and therefore the lines AB and CD are parallel (1); in the same manner it can be demonstrated, if the angles EGB and GHD be given equal.

(1) Prop. pr.

2. Let AGH and CHG be equal to two right angles; and since AGH and BGH are also equal to two right angles, the alternate angles CHG and BGH are equal, and therefore AB and CD are parallel; in the same manner it can be demonstrated, if BGH and DHG be given equal to two right angles.

Lemma 4.

Fig. 15. If any right line NP cut another line BD, it can be produced so as to cut any line XZ parallel to BD.

Take any point O in the line NP: make OS equal to OA (1); draw OM and SE perpendicular to BD, and OC perpendicular to SE, which must fall between S and E, because the angle ASE is acute, since the angle AES is right (2).

(1) Prop. 3. B. 1.
(2) Cor. Prop. 17. B. 1.

In the triangles OMA and SCO, since the sides AM and OC are parallel (3), the angles OAM and SOC are equal, the angles OMA and SCO are also equal, and the sides AO and OS, therefore the sides OM and SC are equal; in the triangles COE, MEO, the alternate angles COE and MEO, CEO and MOE, are equal, as CE and OM, ME and OC, are parallel (3), and the side OE is common to both, therefore the sides EC and OM are equal; but OM and SC are also equal, therefore EC and SC are equal: in the same manner it

(3) Lem. 2.

can be demonstrated, if any equal parts be taken in the line AP, that the increments of the perpendiculars, which are drawn from their extremities to the line BD, are equal, and therefore that these perpendiculars (4) at length become greater than the perpendicular from XZ upon BD, and therefore that the line NP shall cut XZ. (4) Axiom.

AXIOM 12. B. 1.

If a right line EF meet two right lines CD and XZ, and make the internal angles on the same side of it, ZEF and EFD, less than two right angles, these right lines if produced at length must meet upon that side, on which are the angles that are less than two right angles. Fig. 13.

Draw through E a line AB parallel to CD; since the angles ZEF and EFD are less than two right angles, but BEF and EFD are equal to two right angles, BEF is greater than ZEF, therefore XZ cuts the line AB; and therefore the line CD parallel to AB (1); and it must cut CD on the side of the angle ZEF; for if it cut CD on the opposite side, it must cut AB again: which is absurd. (1) Lem. 4.

It must be acknowledged, that this method is not entirely safe from objections; for in the definition it is assumed that a right line can be assigned equidistant in every point from another; and in the demonstrations of lem. 2. prop. 27. and ax. 12. it is assumed, that such a line may pass through every point outside a given line.

PROP. 30. B. 1.

It is evident that two lines, which are parallel to a third, sometimes form but one right line.

COR. 5. PROP. 32. B. 1.

A right angle may be divided also into 6 parts, 12, 24, &c. if each part, into which it is divided by the help of this corollary, be bisected; and so on.

COR. 6, & 7. PROP. 32. B. 1.

Cor. 6. may be demonstrated otherwise, by drawing from any angle B right lines to all the angles except the two adjacent; thus the figure is divided into as many triangles as there are sides except two, and in each triangle the angles are equal to two right angles. But in the case, Fig. 16, the construction cannot be applied, unless the lines be drawn from the point D. The angle EDC is called a reentrant angle.

In this case the external angles, ICD, HBC, GAB, FEA, diminished by the angle EDN, are equal to four right angles; for the value of the external angles taken together is deduced, upon the supposition that each internal angle with its adjacent external is equal to two right angles; but the reentrant angle EDC exceeds two right angles by the angle EDN; therefore the reentrant angle, with the sum of all the other internal and external angles, is equal to twice as many right angles as the figure has sides, and to its excess EDN above two right angles: but the internal angles, with four right angles, are equal to twice as many right angles as the figure has sides; therefore the external angles, ICD, HBC, GAB, FEA, taken together, are equal to four right angles and the angle EDN.

The number of reentrant angles in any figure must be less by three than the number of sides, because the sum of the reentrant angles, must be greater than twice as many right angles as there are reentrant angles; but it must also be less by four right angles than twice as many right angles as the figure has sides; therefore twice the number of sides, deducting four, must be greater than twice the number of reentrant angles; or, the number of sides, deducting two, must be greater than the number of reentrant angles.

PROP. 34. B. 1.

Of the various methods which Clavius has given, to divide a right line into any given number of parts, one should be mentioned here, as it depends upon this proposition.

Let it be required to divide AB into three equal parts; draw AZ, making any angle with it; take any point in it H, make HI and IZ equal to AH, join ZB, and draw IF and HE parallel to ZB (1): the parts of the given line, AE, EF and FB, are equal. Fig. 17. (1) Prop. 31. B. 1.

Draw IG and HO parallel to AB; they are parallel to each other (2), therefore the angles IHO and ZIG are equal (3); ZB and IF are also parallel, therefore the angles IZG and HIO are equal (3), and the sides HI and IZ are equal; therefore the sides HO and IG are equal (4), and therefore EF and FB are equal (5), and so on, if there be a greater number of parts; but AE is equal to EF; for the angles IHO and HAE, HIO and AHE, are equal, and also the sides AH and HI, therefore AE and HO are equal, and therefore AE and EF. (2) Prop. 30. B. 1. (3) Prop. 29. B. 1. (4) Prop. 26. B. 1. (5) Prop. 34. B. 1.

The two following propositions are not undeserving of notice, though they are seldom made use of.

Every quadrilateral figure ABCD, whose opposite sides are equal, is a parallelogram. Fig. 18.

Draw BD: in the triangles BAD, DCB, the sides AB and AD are equal to BC and CD, and BD is common to both, therefore the angles ABD and CDB, ADB and CBD, are equal; therefore the lines AB and CD are parallel, and also AD and BC, and therefore ABCD is a parallelogram.

Every quadrilateral figure, whose opposite angles are equal, is a parallelogram. Fig. 18.

Since A is equal to C, and B to D, A and B together are equal to D and C, therefore the four angles are double of A and B; but the four angles together are equal to four right angles, therefore A and B together are equal to two right angles, and therefore BC and AD are parallel: in the same manner it can be demonstrated that AB and DC are parallel.

PROP. 35. B. 1.

Of the three cases which this proposition admits, and which are noticed in the Arabic version, the Greek copies give only one, Fig. 53. Plate 2.

Having demonstrated the equality of the triangles BEA and CFD, they desire the triangle DOE to be taken away from both, and BOC to be added to the remainders; but these two triangles do not exist in Fig. 54, and 55.

PROP. 40. B. 1.

The equal bases GH and BC are supposed to form one right line.

PROP. 44. B. 1.

Every one acquainted with prop. 45. must be aware that by a *given right line* is to be understood a right line, not only given in quantity, but also in position. This problem, therefore, cannot be solved by producing a side of the triangle, until the produced part be equal to the given line; a method followed by several editors. I can scarcely believe that the construction in the Greek text is Euclid's; for, after having constructed a parallelogram equal to the given triangle, he desires *a side of it to be placed in directum with the given line:* how this direction is to be obeyed I know not; such a removal of figures from place to place is not found among the postulates, nor is it made use of in the construction of other problems. From the figure annexed to this proposition in the Hervagian edition (Fig. 19. Plate 7.) it is most probable that part of Euclid's construction has fallen out of the text; for in it, besides the given triangle, there is another equal to it, and joined to the parallelogram, of which no mention is made in the text: for these reasons I have nearly followed in the construction Campanus's method.

PROP. 46. B. 1.

If perpendiculars were erected at the extremities of the given line, equal to it, the line joining them would

complete the square; this would be an easier construction, but the demonstration would be more tedious, and that alone was regarded by Euclid.

PROP. 47. B. 1.

This proposition is a case of a more universal theorem which is given in the mathematical collections of Pappus.

If parallelograms AEGB and CLHB be described upon the sides AB and CB, of any triangle ABC, and the sides EG and LH, parallel to the sides of the triangle, be produced till they meet in D, these parallelograms taken together are equal to the parallelogram AFIC described upon the remaining side AC, so that IC shall be equal and parallel to the line DB, joining the vertex with the point of concourse. Fig. 21.

Produce DB to K; the parallelograms FK and FABD are upon the same base AF and between the same parallel lines AF and KD, therefore FK is equal to FABD; and the parallelograms FABD and EABG are upon the same base AB, and between the same parallel lines AB and ED, therefore FABD is equal to EABG, and therefore FK and EABG are equal: in the same manner it can be demonstrated, that IK and CLHB are equal: therefore the whole parallelogram AFIC is equal to AEGB and CLHB taken together.

This proposition is not true, in the general form in which Pappus has given it. The parallelograms on the sides must be similarly situated with respect to the triangle; either both on the opposite side from the triangle as in figure 21, or both at the same side with the triangle. If they have different positions with respect to the triangle, then the parallelogram on the base will be equal to the difference of the parallelograms on the sides.

Let EABG and BHLC be the parallelograms on the sides; let AFIC be the parallelogram on the base, whose sides AF and CI are equal and parallel to DB; produce DB till it meet AC in M, and FI till it meet MD in K. Fig. 20.

The parallelogram BHLC is equal to BDIC, because they stand on the same base BC, and between the same parallels; but BDIC is equal to ICMK, because they stand on the same base IC, and between the same parallels; therefore the parallelogram BHLC is equal to the parallelogram ICMK: in the same manner it can be proved, that the parallelogram ABGE is equal to AMKF; but the parallelogram AFIC is equal to the difference between the parallelograms ICMK and AMKF, and therefore is equal to the difference between the parallelograms on the sides BHLC and ABGE.

PROP. 2. B. 2.

Of this and the eight following propositions I have annexed demonstrations in addition to those given by Euclid.

PROP. 6. B. 2.

The fifth and sixth propositions might have been enunciated in the same words, that the rectangle under two lines is equal to the difference between the squares of half their sum and half their difference; as is evident from the figures; for AD and DB, in Fig. 9. may be taken for the given lines, AF and FB, in Fig. 14.

By this proposition we can demonstrate the well known property of lines in arithmetical proportion, that the rectangle under the extremes, together with the square of the common difference, is equal to the square of the mean: for the lines AF, CF and BF, have a common difference CB, since AB is bisected; and the square of CB, together with the rectangle under the extremes AF and BF, is equal to the square of the mean CF.

By this proposition we can also demonstrate a theorem, from which is derived a very useful property of the centre of gravity, that the rectangle under the sum of any number of bodies, and the distance of their centre of gravity from any assumed point, is equal to the sum of the rectangles under each of the bodies and its distance from the same point.

The theorem is, that if a line CB be divided in D, and produced to any point A, the rectangle under the given line CB, and AD the sum of the produced part AC and the adjacent segment CD, is equal to the rectangles under the whole line AB and the segment CD, and under the produced part AC and the other segment DB. Plate 2. Fig. 9.

For the rectangle under AD and CB is equal to the sum of the rectangles under their segments, under AC and CD, CD and CD, CD and DB, and AC and DB; but the rectangles under AC and CD, CD and CD, and CD and DB, are equal to the rectangle under CD and AB, whence it is evident that the rectangle under AD and CB is equal to the rectangles under CD and AB, and under AC and DB.

DEF. 1. B. 3.

This is a theorem, not a definition; and it can easily be demonstrated; for when the centres coincide, if the radii be equal, the circumferences must coincide, whence it follows that the circles and their circumferences are equal.

DEF. 10. B. 3.

If prop. 23. and 26. of this book be compared together, it will appear that this definition is ambiguous; for in prop. 23. it is assumed that similar segments have the angles in one equal respectively to the angles in the other, and in prop. 26. that segments which have one angle in the one equal to one in the other are similar: this ambiguity is removed by prop. 21. in which it is proved that all angles in the same segment are equal; and Euclid does not treat of similar segments before that proposition. These segments are called similar by a sort of anticipation, for proportion is not treated of till book 5: but when Euclid wished to demonstrate the affections of such segments, and was to give them a name in order to avoid a periphrasis, it was better to call them from their relation to their own circles, than to give them a useless name.

PROP. 2. B. 3.

Candalla, whom Tacquet followed, has given a direct proof of this proposition; which Euclid seems to have rejected, because the principle on which it depends, namely, *that the extremity of a right line less than the radius is within the circle*, does not appear among the axioms, and he preferred using an indirect proof to increasing the number of axioms.

PROP. 4. B. 3.

This is a property of every conic section: In the triangle it is evident, for if two lines mutually bisected each other, the sides of the triangle must be parallel, as the alternate angles would then be equal.

PROP. 7. B. 3.

That a right line nearer to that passing through the centre is greater than the more remote, Euclid only demonstrates when they are at the same side; probably, because the other case follows easily from the equality of lines equidistant from the centre; however, I have thought it better to add the demonstration.

PROP. 10. B. 3.

Euclid gives two demonstrations of this proposition; in one, assuming that the circles intersect in three points, he shews that they have a common centre, the intersection of the perpendiculars which bisect the right lines joining the points of intersection; which is contrary to prop. 5. In the other, assuming the same thing, he draws lines from the centre of one of the circles to the points of intersection: these lines are equal, and are drawn to the periphery of the other circle from a point within it; that point is therefore its centre: which also is contrary to prop. 5. But since the first method cannot easily be applied to the case of fig. 15. and the other method cannot at all, as the centre of each circle is without the other, I have thought it better to give a different demonstration.

PROP. 11. B. 3.

The right line must be produced on the side of the centre of the less circle, in order that it may meet the circles at the point of contact.

PROP. 13. B. 3.

In the demonstration of this proposition, Euclid assumes that the right line joining the centres passes through both points of contact, if there be two; which certainly may be demonstrated by prop. 11., but ought not to be assumed here, as in that proposition there is mention made only of one point of contact. Some perhaps may think that there are two points of contact adjacent to each other; and to this case, as R. Simson has remarked, the figure belonging to Euclid's construction is by no means adapted, for the centres of the circles must be placed near the circumferences: I have therefore changed the demonstration, but not for that which Simson has taken from Campanus, as it is liable to the objection that there may be four points of contact; he bisects the line joining the two supposed points of contact by a perpendicular, and says that this must pass through the centres of both circles, and therefore through their points of contact: and it is certainly possible that it might, if the circles touched in four points, at the extremities of the subtense and of the diameter, which Euclid has not elsewhere directly shewn to be impossible.

PROP. 15. B. 3.

R. Simson has given a different demonstration of this proposition, by which he can easily prove the converse of the latter part to be true, namely, that the right line which is greater is nearer to the centre. For the squares of the perpendiculars CF and CV are the complements of the squares of FK and VD, half the inscribed lines, to the square of the radius CK or CD; therefore the greater the squares of the halves, the less the squares of the perpendiculars, and therefore the less the perpendiculars themselves; and *e contra*.

PROP. 16. B. 3.

I have omitted the latter part of this proposition, *that the angle between the diameter and periphery is greater than any rectilineal angle: and that the angle between the periphery and tangent is less than any rectilineal angle,* because it is quite useless, and treats of the comparison of quantities, which will not be universally acknowledged as homogeneous.

SCHOL. 2. PROP. 16. B. 3.

This seems necessary for prop. 34. B. 3. lest any person should imagine that, if the angle V be indefinitely small, the right line, making with the tangent an angle equal to it, might fall entirely without the circle.

PROP. 20. B. 3.

In the demonstration of this proposition as it now stands in the Greek text, two principles are assumed, *that if two magnitudes be each double of two others, the sum of the former is double the sum of the latter;*—and *if two quantities be each double of two others, that the difference of the two former is double of the difference of the two latter.* Clavius asserts, that Euclid had a right to assume these principles, since he has demonstrated them in prop. 1. and 5. B. 5. a general theorem not depending upon this proposition, of which these two principles are particular cases; I have thought it better to change the demonstration a little, than to make such a defence; and to the demonstration as now given I hope no objection can be made: since the sum of the parts is equal to the whole, and magnitudes, which are double of equals, are themselves equal, we may justly assert that the sum of quantities which are double of the parts, is double of the whole, and *e contra:* but if any are dissatisfied with this method, they may demonstrate the proposition with Clavius thus:

Fig. 22. Let the angle at the centre fall within the angle at the periphery; and draw through their vertices the line BE.

The angles CAB and CBA are equal (1), and also CBD and CDB (1); therefore the angles CAB and CDB taken together are equal to ABD (2); therefore the sum of the three angles CAB, ABD and CDB, is double of ABD; but the angle ACE is equal to the angles CAB and CBA (3), and the angle ECD is equal to the angles CDB and CBD (3), and therefore the angle ACD is equal to the sum of the angles CAB, ABD and CDB; but the sum of these angles is double of the angle ABD, and therefore the angle ACD is double of ABD.

(1) Prop: 5. B. 1.
(2) Ax. 2.
(3) Prop. 32. B. 1.

If either side CD, of the angle at the centre, be cut by a side BA of the angle at the periphery, draw CB; since the angles CBD and CDB are equal (1), and CBD is greater than EBD, CDB is greater than EBD; at the point D, and with the line DB, make the angle BDF equal to the angle EBD (4); if these equals be taken away from the equal angles CDB, CBD, the remaining angles CDF and CBF are equal; but CBF and CAB are equal (1), therefore CDF is equal to CAB; the vertically opposite angles CEA and DEF are also equal (5), therefore the remaining angle ACE in the triangle AEC is equal to EFD in the triangle DFE (6); but the angle EFD is equal to the sum of the angles FBD and FDB (3), which are equal to each other; therefore EFD is double of ABD, and therefore ACD is double of ABD.

Fig. 23.
(4) Prop. 23. B. 1.
(5) Prop. 15. B. 1.
(6) Cor. 2. Prop. 32. B. 1.

PROP. 25. B. 3.

This method of finding the centre of the circle, which is elsewhere made use of by Euclid, is substituted for the prolix construction which he has employed in this proposition.

PROP. 26. B. 3.

The second part of this proposition, when the given angle is not acute, is demonstrated differently by Euclid; he merely says, *take away the equals ABC and DEF from the equal circles, and the remaining arches AC and DF are equal.* But if the given angle be right, this proof is inconclusive; for the opposite angle is also

right, and the demonstration of the first part is inapplicable to a right angle.

PROP. 31. B. 3.

To the enunciation of this proposition are added in the Greek text the following words: *and the angle of a greater segment is greater than a right angle; and the angle of a less segment is less than a right angle.* I have omitted this part of the proposition for the reasons assigned in N. to prop. 16. B. 3. nor can I believe that these parts of prop. 16. and 31. were ever written by Euclid: for the principles, on which the demonstration of the latter depends, are subversive of those by which the former is proved; as must be evident to any person considering the subject.

PROP. 33. B. 3.

In the Greek text the construction of the second case is different, AB being bisected, and a perpendicular erected at the point of bisection; but by this construction the demonstration is rendered more prolix. It is evident from Ax. 12. that the lines AO and BO must meet, for each angle OAB and OBA is less than a right angle.

DEFINITIONS OF BOOK 4.

I have omitted two definitions which are prefixed to this book in the Greek text, those of a rectilineal figure inscribed in a rectilineal figure, and circumscribed about a rectilineal figure, because they are useless; as Euclid does not treat of such inscription or circumscription.

PROP. 3. B. 4.

The tangents ML and MN must meet LN; for, if AB and BC be drawn, the angles towards L and N are less than two right angles; and since the two angles at D together with those at F are equal to four right angles, and the internal angles EDF and EFD are less

than two right angles, EDG and EFH are greater than two right angles; therefore, if AKB and AKC, which are equal to EDG and EFH, be taken away from the four angles at K, it is evident that the lines KB and KC make an angle at the side of M; and therefore, if BC be drawn, the angles between it and the tangents at the side of M are less than two right angles, and therefore the tangents meet at that side.

PROP. 4, & 5. B. 4.

It is evident, from ax. 12. that the right lines bisecting the angles of the triangle in prop. 4. must meet; and from the same axiom it appears that, if DE be drawn, the perpendiculars DF and EF in prop. 5. must also meet.

PROP. 6. B. 4.

Euclid demonstrates that the inscribed figure is equilateral, by prop. 4. B. 1. whence it is evident that the bases AB and AD of the triangles AEB and AED are equal, and so on of the rest: but the demonstration from the equality of the arches seems to me better.

PROP. 7. B. 4.

This proposition, as it now stands in the Greek copies, is corrupt; for in the demonstration are inserted the words, *that FH is a parallelogram, and therefore that FG and KH, also FK and GH are equal,* which are totally useless, as the demonstration proceeds to shew that FG and KH are each equal to AC, and that FK and GH are each equal to BD; whence it follows that FG, GH, HK and KF, are equal to one another. A similar error has crept into the eighth proposition, where it is said, *that GD, BK, GA, AH and AD, are parallelograms;* which is not necessary to the demonstration.

PROP. 11. D. 4.

If the division of an angle into any number of parts be conceded, and the construction of an isosceles triangle, in which the angles at the base may be any multiples of the vertical angle, any figure of an odd number of sides can be inscribed in a circle; and if the angles at the base be the sesquialteral multiples of the vertical angles, figures of an even number of sides may be thus inscribed. Subtract unity from the number of sides, divide the remainder by two, and construct an isosceles triangle, in which the angles at the base are to the vertical as the quotient to unity; and by this triangle the required figure can be inscribed. *Ex. gr.* if a heptagon were to be inscribed, the angle at the base ought to have to the vertical angle the ratio of three to one: if an octagon, as $3\frac{1}{2}$ to 1.

COR. 5. PROP. 15. B. 4.

Hence it is evident, that the side of an equilateral triangle is incommensurable with the diameter of the circumscribing circle, for it has to the diameter the ratio of the square root of three to two.

PROP. 16. B. 4.

The circumference of a circle, which by this proposition is divided into fifteen parts, can, by bisecting each of these parts (1), be divided into thirty parts, and again into 60, 120, &c. And also, if each of the four parts into which a circle is divided by prop. 6, be bisected, a circle can be divided into 8, and again into 16. 32, &c. In like manner, by prop. 15, it can be divided into 3, 6, 12, &c.; and by prop. 11. into 5, 10, 20, &c. But a geometrical method of dividing a circle into any given number of parts has never yet been discovered.

(1) Prop. 13. B. 3.

A celebrated theorem of Proclus concerning ordinate polygons should not be omitted: the theorem is, that only three ordinate polygons can be so placed at

a point, as to make a continuous surface; for this purpose it is necessary that the angle of the polygon should be an aliquot part of four right angles, since the angles at any point are equal to four right angles; therefore, as the angle of an equilateral triangle is the sixth part of four right angles, the angle of a square the fourth part, and the angle of an hexagon the third part, it is evident, that six equilateral triangles, four squares, or three hexagons can be so placed at a point as to make a continuous surface. But the angle of a pentagon is greater than the fourth part, and less than the third part, of four right angles; and the angle of any other ordinate polygon is greater than the third part of four right angles, and therefore they cannot be so placed at a point, as to form a continuous surface.

DEF. 4. B. 5.

By this definition Euclid determines what magnitudes he acknowledges as homogeneous; in general, quantities are considered homogeneous, which he considers heterogeneous; *ex. gr.* a finite and an infinite line, the angle of contact and a rectilineal angle.

DEF. 5. B. 5.

How proportional magnitudes ought to be defined, is still a subject of controversy among geometers. Euclid defines them thus: *The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equi-multiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth being taken, the equi-multiples of the first and third either together exceed, are equal to, or are less than the equimultiples of the second and fourth.* This definition is liable to the objection, that there is not the least resemblance between it and the common notions of similitude or equality of ratios: it must be confessed indeed, that Euclid has demonstrated that magnitudes thus related retain the same properties, however they may be inverted or subjected to the various changes

made use of by geometers; but this is not sufficient: the connexion ought to be shewn between this definition and that relation of two ratios to which the name of equality is commonly given. Against this objection Barrow ably argues in his mathematical lectures, and after having overturned the various theories invented since the time of Euclid, endeavours to put a stop to the controversy, and repress every attempt towards forming a new theory, in the following words: *Per has naturas* (scil. magnitudinum proportionalium) *nil aliud intelligi quam rei definitæ nomini, quatenus in usu communi versatur, respondentes conceptus aut significatus aliquos imperfectos et indistinctos, in scientiis minime respiciendos, ad quos proinde nullatenus exigendæ sunt definitiones; imo secludendis et eliminandis iis, ipsorumque loco substituendis rerum certis distinctis atque claris ideis, efformantur definitiones.* If this be true, Euclid's definition is certainly the best, for undoubtedly it is wholly foreign from common use: but it is evident from the work itself, that Euclid never formed the definitions of his Elements by such a rule; and even Barrow acknowledges (Lect. 6. A. D. 1664.), *Quod expediat a facilioribus magisque familiaribus passionibus argumentandi initium sumere, seu subjectum definire.* Here it is confessed, that, if a more familiar definition than Euclid's can be given, it ought to be preferred, provided all the properties of proportionals which are necessary can be demonstrated from it: that the definition I have given is of this nature, will, I hope, be acknowledged: borrowed from the common idea of proportion, it is nevertheless accurate, and from it all the properties of proportionals, including even Euclid's definition, are demonstrated.

That the young student may more easily understand this definition, I annex some examples of equal and unequal ratios. Let the ratio of 6 to 42, and of 9 to 63, be compared together; and as often as a submultiple, 1, 2, 3 (or any other), of the first antecedent is contained in the first consequent, so often the equi-submultiple $1\frac{1}{2}$, 3, $4\frac{1}{2}$ (or any other), of the latter antecedent is contained in its consequent.

Likewise if the given ratios be that of 8 to $\sqrt{\ }128$

and that of 16 to $\sqrt{512}$, any submultiple of the latter antecedent is contained as often in its consequent, as the equi-submultiple of the first antecedent is contained in its consequent. Therefore in both cases the four quantities compared are proportional. But if the given ratios be that of 8 to 13, and of 16 to 27, although the submultiples, 2, 4, 8, of the latter antecedent are contained as often in the consequent, as the equi-submultiples 1, 2, 4, of the first antecedent are contained in its consequent, yet, if the equi-submultiples of the antecedents 1 and $\frac{1}{2}$ be taken, the former shall be oftener contained in 27 than the latter in 13; therefore the ratios of 16 to 27, and of 8 to 13, are not equal; but the first antecedent has a less ratio to its consequent than the latter has to its consequent.

DEF. 7. B. 5.

This definition is substituted for the following: *When of the equi-multiples of four magnitudes the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth: then the first is said to have to the second a greater ratio than the third has to the fourth; and on the contrary the third is said to have to the fourth a less ratio than the first has to the second.*

DEF. 12. B. 1.

The definition of compound ratio is generally given among the definitions of the sixth book; but R. Simson, having proved that it had been corrupted, restored it to its original form and place, and thus explained the purpose for which it had been introduced into geometry: *The use of compound ratio consists wholly in this, that by means of it circumlocutions may be avoided, and thereby propositions may be more briefly either enunciated or demonstrated, or both may be done; for instance, if prop. 23. of the 6th book were to be enun-*

ciated without mentioning compound ratio, it might be done as follows: If two parallelograms be equiangular, and if as a side of the first to a side of the second, so any assumed straight line be made to a second straight line; and as the other side of the first to the other side of the second, so the second straight line be made to a third; the first parallelogram is to the second as the first straight line to the third: and the demonstration would be exactly the same as we now have it. But the ancient geometers, when they observed this enunciation could be made shorter by giving a name to the ratio which the first straight line has to the last, by which name the intermediate ratios might likewise be signified, of the first to the second, and of the second to the third, and so on if there were more of them, they called this ratio of the first to the last, the ratio compounded of the ratios of the first to the second, and of the second to the third straight line; that is, in the present example, of the ratios which are the same with the ratios of the sides: and by this they expressed the proposition more briefly.

The three remarkable species of compound ratios have been called duplicate ratio, triplicate ratio, and sesquialteral or sesquiplicate ratio. The former two have been defined by Euclid; the latter deserves some explanation, as, though not used in geometry, it is frequently mentioned in astronomy.

If there be three magnitudes proportional, and also four others continually proportional, and the first be to the last in the one series as the first to the last in the other, the ratio which the first has to the second in the series of three proportionals is said to be sesquialteral or sesquiplicate of the ratio which the first has to the second in the series of four proportionals.

For in the series of four proportionals find a mean between the first and last; the first is to this mean, as the first in the series of three proportionals to the second (1); but the former ratio is composed of the ratios of the first to the second in the series of four proportionals, and of the second to the mean, and the former of these ratios is evidently duplicate of the other; and hence the compound ratio is called sesquialteral of the simple.

(1) Prop. 39. B. 5.

Let the series be	2	:	16	:	128	
And	1	:	4	:	16 :	64

The ratio of 2 to 16 is sesquialteral of the ratio of 1 to 4: find a mean proportional 8 between 1 and 64: 1 is to 8 as 2 to 16; but the ratio of 1 to 8 is compounded of the ratios of 1 to 4 and of 4 to 8, the former of which is duplicate of the latter.

DEF. 14. B. 5.

These different changes of proportional quantities will perhaps be better understood by the assistance of the following table:

Let the given ratio be	A : B :: C : D. Then
By alternation -	A : C :: B : D.
By inversion -	B : A :: D : C.
By composition -	A+B : B :: C+D : D.
By division -	A—B : B :: C—D : D.
By conversion -	A : A + B :: C : C + D.
or -	A : A — B :: C : C — D.
And if there be three magnitudes -	A, B and C.
And other three -	D, E and F.
And there should be	A : B :: D : E,
and -	B : C :: E : F, then
Ex æquo ordinate	A : C :: D : F,
But if -	A : B :: E : F,
and -	B : C :: D : E, then
Ex æquo perturbate	A : C :: D : F.

AX. 1. B. 5.

Euclid makes no mention of submultiples; but it is evident that all the affections of multiples, which he places among the axioms, must also belong to submultiples.

PROP. 1. B. 5.

Although this proposition and some others might be assumed as axioms, I preferred demonstrating them to encreasing the number of axioms.

COR. 1. PROP. 7. B. 5.

The word *contained* is ambiguous; for a magnitude, which is contained in another, may either be a submultiple of it or not, and from this ambiguity arises the difficulty of many propositions in this book.

PROP. 32. B. 5.

This proposition is Euclid's definition of proportionals.

DEF. 4. B. 6.

This and the two following definitions are added, as necessary for understanding prop. 27, 28, and 29.

PROP. 1. B. 6.

Euclid's demonstration is different, being founded on his definition of proportional magnitudes.

PROP. 3. B. 6.

Fig. 24. If, the side BA being produced, the external angle CAF be bisected, and if the bisecting line meet the base, it can be shewn in the same manner that the segments of the base are proportional to the other sides, and *e converso.* The proposition appears to me to have formerly been so expressed as to include both cases; for now it begins thus: "If the angle of a triangle be bisected, and the *bisecting line should cut the base;*" now the line bisecting the internal angle must cut the base, and therefore Euclid would have expressed the proposition absolutely, if he had not had in view the bisection of the external angle.

The lines bisecting the internal and external angles divide the base produced in *harmonical* proportion, that is, so that the first is to the third as the difference between the first and second to the difference between the second and third.

BE is to EC as BD is to CD, therefore by alternation BE is to BD as EC is to CD; but EC is the difference between BE and BC, and CD is the difference between BC and BD; therefore BE, BC and BD, are *harmonically* proportional.

PROP. 6. B. 6.

In this proposition, and also in the preceding, the construction, which in Euclid is definite, is expressed indefinitely; lest any one should imagine there was but one side of the triangle on which the construction could be made. Euclid says, *construct at the side DF*, &c. as if it could not be constructed at the other side.

PROP. 11. B. 6.

If the given be a ratio of less inequality, LO to LR, the series can be continued until a magnitude be found greater than any assigned. Fig. 25.

For let LO, LR, LQ and LI, be continually proportional, since LO is to LR as LR to LQ, by conversion, LO is to OR, as LR to RQ; but LR is greater than LO, therefore RQ is greater than OR (1): in the same manner it can be shewn that IQ is greater than RQ; since therefore quantities continually encreasing are added to the first, a magnitude can be found greater than any assigned. (1) Prop. 23. B. 5.

If the given be a ratio of greater inequality, AB to CB, the series can be continued until a magnitude be found less than any assigned. Fig. 26.

Let the assigned quantity be OL: as CB is to AB so let OL be to LR, and continue the series till IL be found greater than AB; continue the ratio of AB to CB through as many terms, and let the last be FB; FB is less than OL.

For since there are two series of magnitudes proportional and equal in number, *ex æquo* AB is to FB as IL to OL; but AB is less than IL (1), therefore FB is less than OL (2); and in the same manner a magnitude can be found less than any other given one.

(1) Constr.
(2) Prop. 23. B. 5.

Tacquet has given from Gregorius a S. Vincentio, the following method of finding a series of lines in any given ratio of greater inequality, and of exhibiting the sum of the series continued through an infinite number of terms.

Fig. 27.

Take in any right line AZ, parts AB and BC equal to the terms of the given ratio, draw AL and BO perpendicular to AZ and equal to AB and BC; draw LO meeting AZ in Z, through C draw CQ perpendicular to AZ, and CQ is the third proportional to AB and BC: take CE equal to CQ, the perpendicular ER is the fourth proportional, and so on; and AZ is equal to the sum of the series if it be continued through an infinite number of terms.

(1) Schol. Prop. 4. B. 6.
(2) Constr.
(3) Prop. 18. B. 5.

Part 1. AZ is to BZ as AL to BO (1), or as AB to BC (2), by alternation AZ is to AB as BZ to BC, by conversion AZ is to BZ as BZ to CZ, but AZ is to BZ as AL to BO (1), and BZ is to CZ as BO to CQ (1); therefore AL is to BO as BO to CQ (3): and in the same manner it can be proved that the other perpendiculars are proportional.

(4) Prop. 32. B. 5.

Part 2. Since AL is to BO as AZ to BZ, and AL is less than AZ, BO is also less than BZ (4); likewise CQ, ER, FS, &c. are less than CZ, EZ, FZ, &c. but AL is equal to AB, BO to BC, CQ to CE, &c. therefore the whole series of proportionals is not greater than AZ. And since AZ, BZ, CZ, &c. are continually proportional, the last term must be less than any assigned magnitude, and therefore the sum of AB, BC, CE, &c. or the whole series of proportionals, if the number of terms be infinite, is not less than AZ.

Fig. 27.

The difference between the first and second terms,

the first term, and the sum of the series, are continually proportional. For LX is to XO as LA to AZ (1); but LX is the difference between LA and OB (2), and XO is equal to AB, and therefore equal to AL (2), and AZ is the sum of the series.

(1) Cor. 1. Prop. 4. B. 6.
(2) Constr. & Prop. 34. B. 1.

PROP. 13. B. 6.

Plato, Philo of Byzantium, and Des Cartes, invented the following methods of finding two mean proportionals, which though not strictly geometrical, are yet deserving of notice.

Plato's method.

Insert in the side of a square a ruler moveable at pleasure along that side, and always perpendicular to it; place the given right lines AB and CB at a right angle, and produce them to Z and X: then apply the angle of the square to BX, so that, when its side passes through A, and the ruler through C, the vertex of the angle between the ruler and the side of the square in which it is inserted may be somewhere in the line BZ; then BD and BE are the two required mean proportionals. For in the right-angled triangle ADE, DB is drawn from the right angle perpendicular to the opposite side, therefore AB is to DB as DB to BE (1); and for the same reason DB is to BE, as BE to BC.

Fig. 28.
(1) Cor. Prop. 8. B. 6.

Philo's method.

Place the given right lines AB and BC at a right angle, and complete the rectangle BADC; circumscribe a circle about it, and produce DC and DA; then apply a ruler at the point B, so that the parts of it FO and BG shall be equal: then AF and CG are the two means sought.

Fig. 29.

(1) Constr. Since GB is equal to FO (1), GO is equal to BF, and therefore the rectangle under OG and GB is equal to the rectangle under BF and FO; but the rectangle under OG and GB is equal to the rectangle under DG
(2) Cor. Prop. 36. B. 3. and GC (2), and the rectangle under BF and FO is equal to the rectangle under DF and FA (2); therefore the rectangle under DG and GC is equal to the rectangle under DF and FA, and therefore DG is to DF
(3) Prop. 14. B. 6. as AF to GC (3); but on account of the parallel lines
(4) Cor. 1. Prop. 4. B. 6. DG and BA, DG is to DF as AB to AF (4); and on account of the parallels DF and CB, DG is to DF as
(5) Prop. 18. B. 5. GC to CB (4), and therefore BA is to AF as AF to GC, and AF to GC as GC to CB (5).

Des Cartes's method.

Let there be two rulers AZ and AX, moveable on a pivot at the point A, and in these let there be alternately inserted perpendicular rulers BC, CD, DE, &c. so that in opening the first rulers they shall push each other forward; let the given lines be AB and AE; apply the first perpendicular BC at the point B, and open the rulers until the third perpendicular DE pass through the point E: AC and AD are the two means sought.

Since in the triangle ACD the angle ACD is right, and CB is a perpendicular from it upon the opposite
(1) Cor. Prop. 8. B. 6. side, AB is to AC as AC to AD (1); likewise in the right-angled triangle ADE, AC is to AD as AD to AE; therefore AC and AD are mean proportionals between the given lines AB and AE.

Fig. 31. If four mean proportionals are to be found, open the rulers till FG the fifth perpendicular pass through E; and so on if there be six or eight, &c.

But if the number of means required be odd, the given lines AB and AE must be applied to the same ruler; as is evident from fig. 31. where between AB and AE, there are three mean proportionals, AC, AD, and AF.

COR. 2. PROP. 16. B. 6.

The second and third corollaries are cases of the same general theorem, that, if from the angle of any triangle inscribed in a circle two lines be drawn, one to the opposite side, the other cutting off from the circle a segment containing an angle equal to that made by the first drawn line and the side of the triangle which it meets, the rectangle under the sides of the triangle is equal to the rectangle under these lines.—In cor. 2. these lines coincide, as is evident.

COR. 1. PROP. 20. B. 6.

That the two polygons should be in the duplicate ratio of the first given line to the second, these lines must be between the equal angles, and the polygons on them be similarly described. Upon a given line as many polygons can be constructed of different magnitudes, and yet similar to a given polygon, as there are sides of different lengths in the given figure: let the given figure be a quadrilateral, whose sides are proportional to the numbers 1, 2, 4, and 6, and let it be required to construct another upon a line equal to the greatest of these; if it be constructed, so that the adjacent angles may be equal to those adjacent to the equal side in the given figure, the quadrilaterals are equal; but if, so that they may be equal to the angles adjacent to the sides denoted by 4, or 2, or 1, the figures are to each other as 36 to 16, or as 36 to 4, or as 36 to 1.

PROP. 22. B. 6.

In the demonstration of the second part of this proposition, Euclid assumes that the lines, upon which equal polygons similar and similarly posited are described, are equal; I have preferred demonstrating in B. 5. that the subduplicate ratios of equal ratios are

equal, whence the demonstrations of the two parts are similar.

PROP. 23. B. 6.

This demonstration is taken from Candalla. Euclid assumes lines proportional to the sides of the parallelograms, as is done in cor. 1. which does not seem necessary for the demonstration of the proposition.

PROP. 27. B. 6.

From this proposition is obtained the solution of a problem, which seems to be beyond the limits of elementary Geometry, namely, to find the greatest parallelogram which can be inscribed in a given triangle.

Fig. 32. Since the defects of the parallelograms AI, AG, &c. are parallelograms similar to EF, they are about the same diagonal BZ, and are therefore inscribed in the given triangle AZB; of these the greatest is AG described upon half the base, whose sides therefore bisect the other sides AZ and BZ of the triangle.

PROP. 32. B. 6.

Fig. 33. The words, *and that the sides which are not homologous should form the angle at which they are placed,* are added to the enunciation of this proposition, which otherwise is not true. Candalla justly remarks, that the conclusion does not follow from the conditions in the Greek text, as is evident from fig. 35. in which the homologous sides AB and DC, BC and ED are parallel, but the other sides do not form one right line; for this it is necessary that the angles contained by the proportional sides should have a common alternate angle, in order that they may be proved equal. In fact, this proposition contains two conditions; for there are triangles which have two sides of the one proportional to two in the other, and yet which cannot be so placed as

to have these sides parallel; that this may be done, the angles contained by them must be equal.

PROP. 33. B. 6.

Euclid demonstrates this proposition differently, from his own definition of proportionals.—His method is liable to many difficulties; for the given angles may be right or obtuse, whose multiples are therefore not angles, according to his own definition of an angle; and if in fig. 34. the angles ACD and ACB be given, one a right angle, the other half a right angle, they can be so multiplied (by 4) that the former becomes the right line CA, the latter the right line AG: and this is one cause for changing the fifth definition of Book 5.

FINIS.

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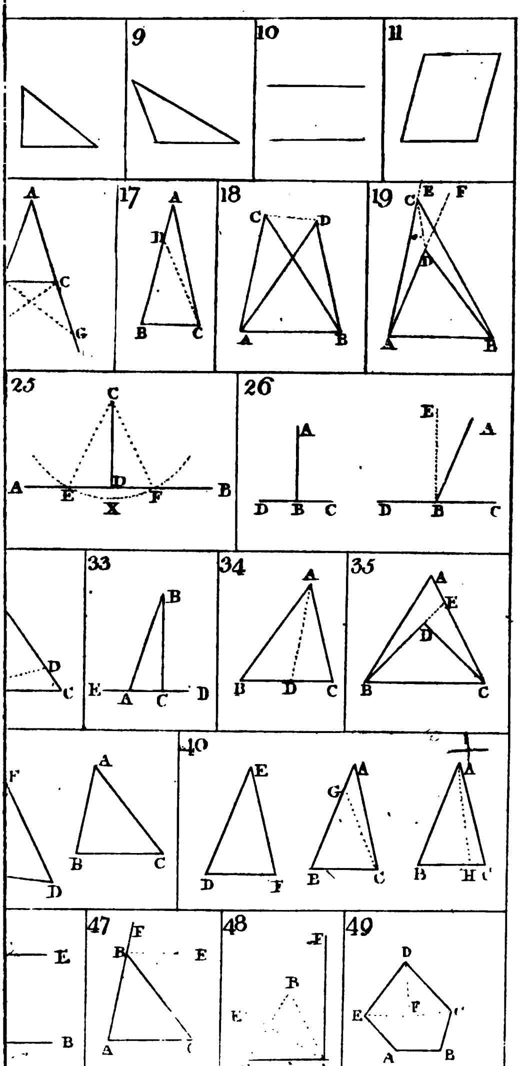

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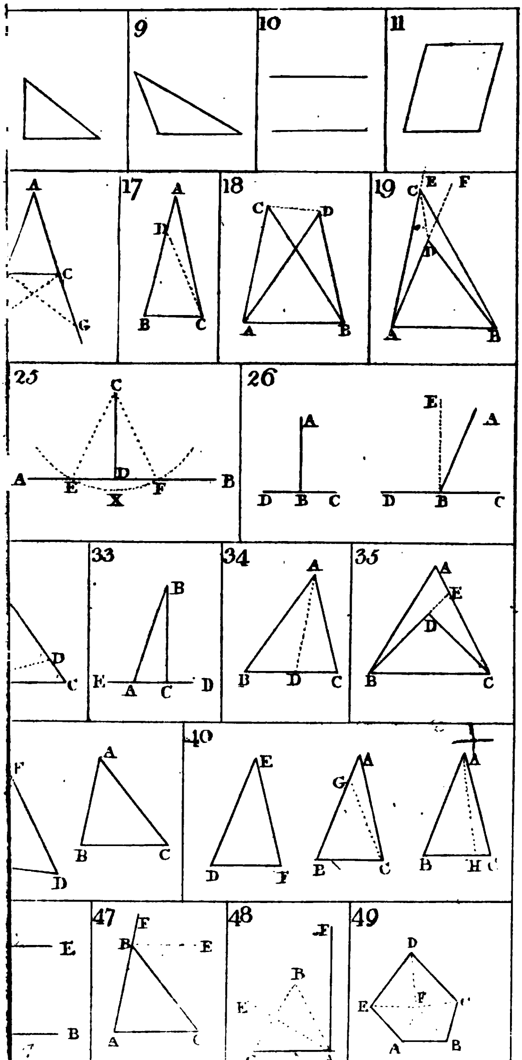
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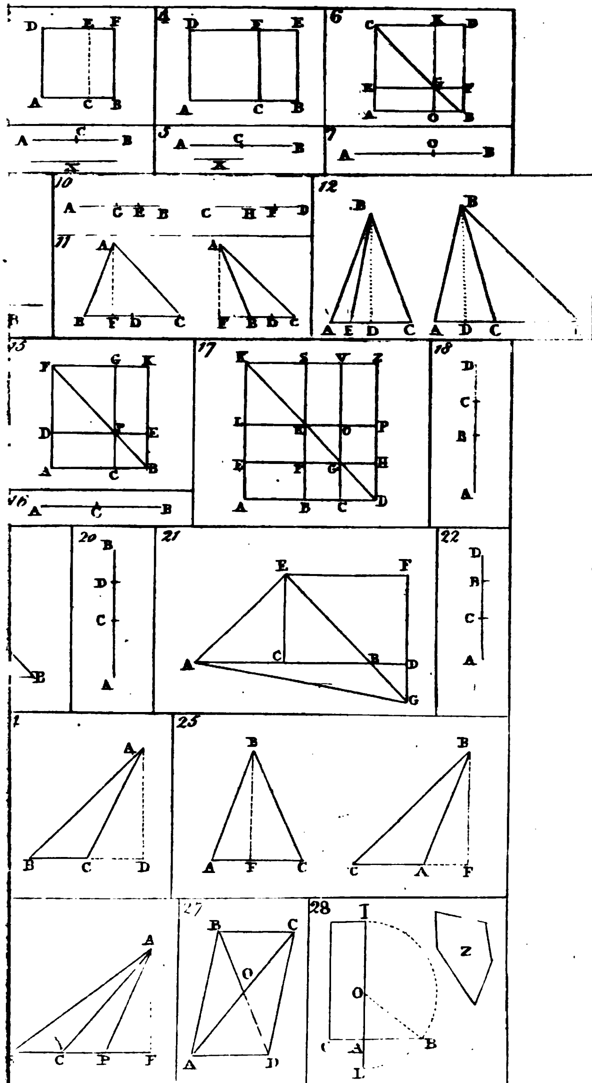

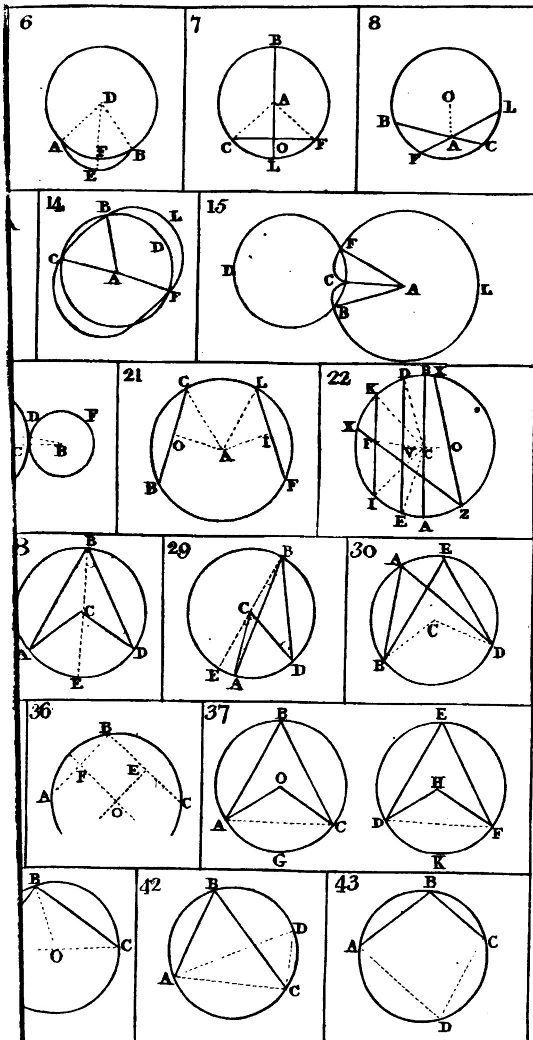
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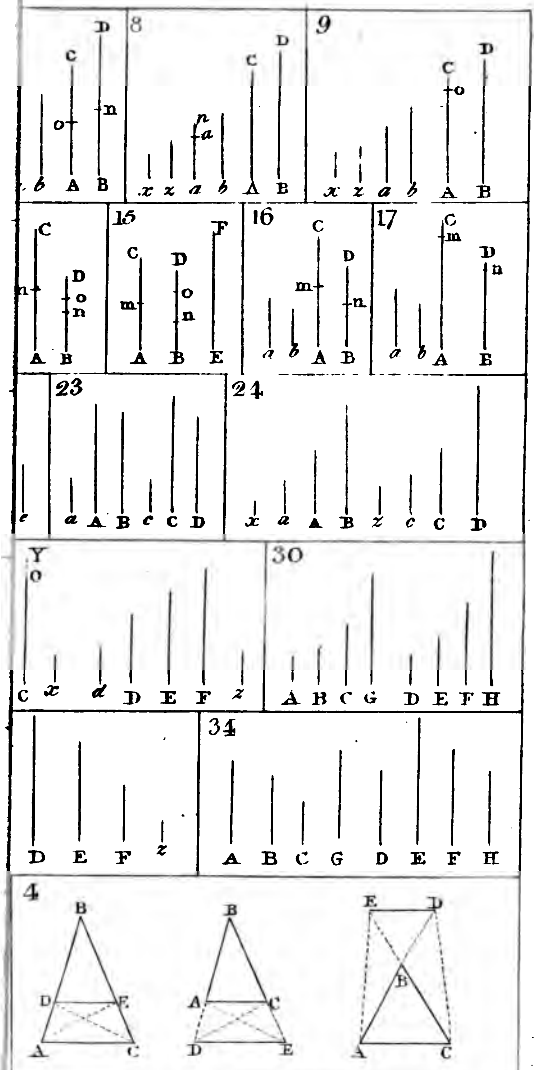

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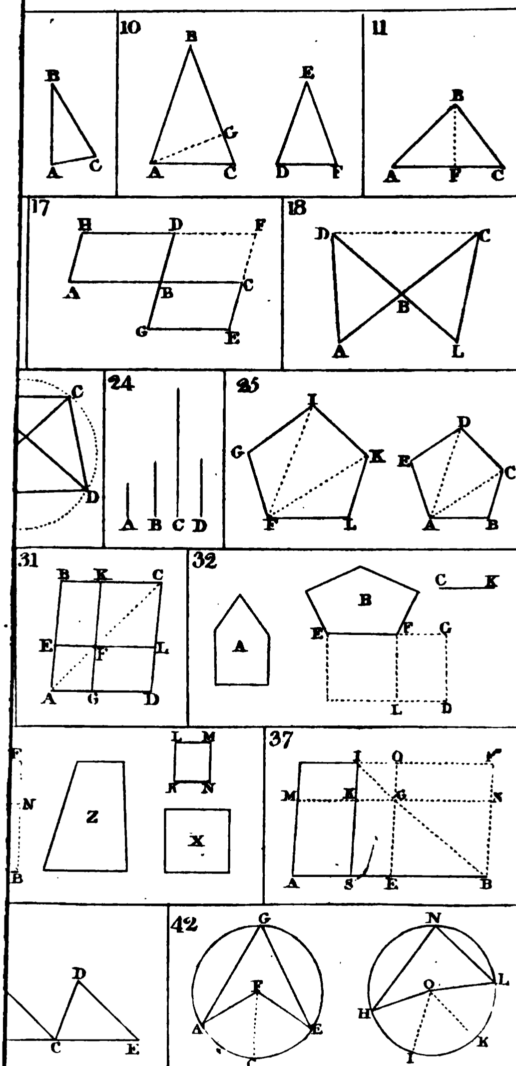
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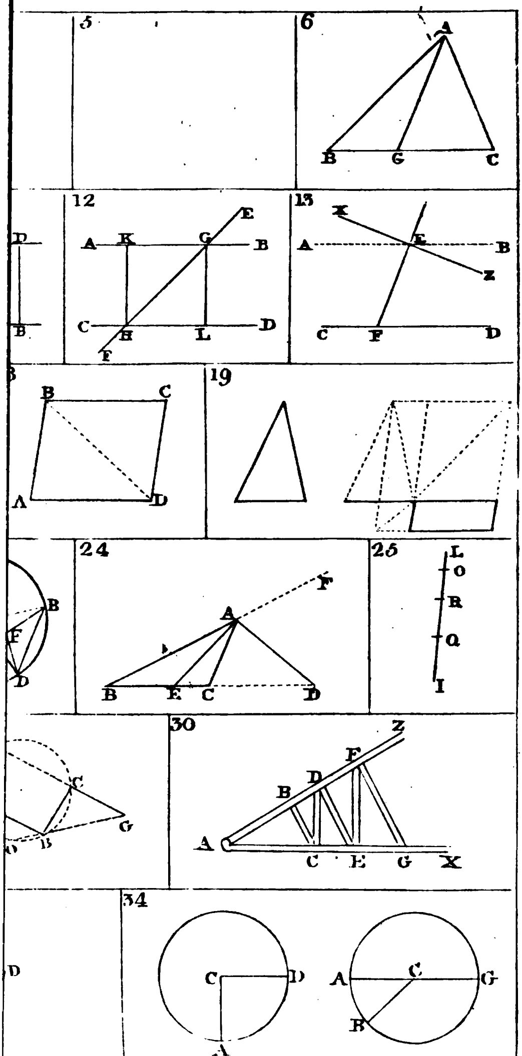
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